HOMOGENEOUS 8-MANIFOLDS ADMITTING INVARIANT Spin(7)-STRUCTURES

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ABSTRACT. We study compact, simply connected, homogeneous 8-manifolds admitting invariant Spin(7)-structures, classifying all canonical presentations G/H of such spaces, with G simply connected. For each presentation, we exhibit explicit examples of invariant Spin(7)-structures and we describe their type, according to Fernández classification. Finally, we analyse the associated Spin(7)-connection with torsion.

1. INTRODUCTION

A Spin(7)-structure on an 8-manifold M is characterized by the existence of a 4-form Φ which can be pointwise written as

$$\Phi|_x = e^{0123} + e^{0145} - e^{0167} + e^{0246} + e^{0257} + e^{0347} - e^{0356}$$
$$+ e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} - e^{1247} + e^{1256},$$

for some basis $(e^0, \ldots, e^7)$ of the cotangent space $T_x^* M$, where e^{ijkl} denotes the wedge product of covectors $e^i \wedge e^j \wedge e^k \wedge e^l$. Any such form is called *admissible*, and it gives rise to a Riemannian metric g_Φ and an orientation on M by the inclusion Spin(7) $\subset$ SO(8) (cf. [16]). The existence of Spin(7)-structures depends on the topology of the manifold [21]. In particular, M has to be orientable and spin.

By [12], the intrinsic torsion of a Spin(7)-structure can be identified with the exterior derivative of the corresponding 4-form Φ. As a consequence, the Riemannian holonomy group $\mathrm{Hol}(g_\Phi)$ is a subgroup of Spin(7) if and only if $\mathrm{d}\Phi = 0$. In such a case, the metric g_Φ is Ricci-flat, and the Spin(7)-structure is said to be *torsion-free*. More generally, the decomposition of the Spin(7)-module $\Lambda^5((\mathbb{R}^8)^*)$ into irreducible submodules allows one to divide Spin(7)-structures into four classes, which were first described in [12]. Recently, a description of these classes in terms of spinorial equations has been obtained in [23].

Since Φ is parallel with respect to the Levi Civita connection ∇^{g_Φ} of g_Φ if and only if $\mathrm{Hol}(g_\Phi) \subseteq$ Spin(7), any other linear connection on M preserving the Spin(7)-structure Φ must necessarily have torsion. By [15], on (M, Φ, g_Φ) there exists a unique connection with totally skew-symmetric torsion T preserving both Φ and a nontrivial spinor. It is given by $\nabla := \nabla^{g_\Phi} + \frac{1}{2}T$.

The aim of this article is to study compact, simply connected 8-manifolds endowed with a Spin(7)-structure and acted on (almost) effectively and transitively by a compact connected Lie group G of automorphisms. Every such manifold admits a presentation of the form $(M = G/H, \Phi)$, where H is a closed subgroup of G, and Φ is a G-invariant admissible 4-form. From the algebraic point of view, $M = G/H$ is a compact, simply connected, (almost) effective homogeneous space whose isotropy action on the tangent space is equivalent to the action of a closed subgroup of Spin(7) on $\mathbb{O} \cong \mathbb{R}^8$.

Recall that every compact, simply connected, homogeneous space M admits a *canonical presentation*, that is, it can be written as $M = G/H$ with G a compact, connected, simply connected, semisimple Lie group and H a closed connected subgroup of G (see e.g. [8, 28]). All compact, simply connected, homogeneous 8-dimensional manifolds G/H of a compact, connected, simply

connected Lie group G were classified in [18]. Moreover, a classification of compact simply connected Riemannian symmetric spin manifolds was given in [11]. A topological examination of these spaces allows us to obtain the following.

Theorem A.

a) *The canonical presentations of all compact, simply connected, non-symmetric almost effective homogeneous spaces admitting a $\mathsf{Spin}(7)$-structure are exhausted by*

  1) $\dfrac{\mathsf{SU}(3)}{\{e\}}$;

  2) $C_{k,\ell,m} := \dfrac{\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)}{\mathsf{U}(1)_{k,\ell,m}}, \quad k \geq \ell \geq m \geq 0, \ k > 0, \ \gcd(k,\ell,m) = 1$;

  3) $\dfrac{\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)}{\Delta\, \mathsf{SU}(2)} \times \dfrac{\mathsf{SU}(2)}{\mathsf{U}(1)}$;

  4) $\dfrac{\mathsf{SU}(3)}{\mathsf{SU}(2)} \times \mathsf{SU}(2)$.

  As smooth manifolds, the spaces a2) and a3) are diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$, while the space a4) is diffeomorphic to $\mathbb{S}^5 \times \mathbb{S}^3$.

b) *The compact, simply connected, symmetric spaces admitting a $\mathsf{Spin}(7)$-structure are exhausted by $\mathsf{SU}(3)$, $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^5 \times \mathbb{S}^3$, $\mathbb{HP}^2$ and the exceptional Wolf space $\dfrac{\mathsf{G}_2}{\mathsf{SO}(4)}$.*

The manifold $C_{k,\ell,m}$ appearing in the above theorem is a torus bundle over the homogeneous Kähler-Einstein manifold $(\mathsf{SU}(2)/\mathsf{U}(1))^{\times 3}$, and hence a non-Kähler C-space. Invariant Einstein metrics on it were discussed in [8, 34]. When $m = 0$, $C_{k,\ell,0}$ is the direct product of $\mathbb{S}^3$ with the total space of a circle bundle over $\mathbb{S}^2 \times \mathbb{S}^2$. Furthermore, $C_{1,0,0} = \mathsf{Spin}(4) \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)}$.

On the other hand, the homogeneous space $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$ is an example of a Calabi-Eckmann manifold. This is a torus bundle over $\mathbb{CP}^2 \times \mathbb{CP}^1$, and hence also a non-Kähler C-space. By Jensen [13], the 5-sphere $\mathsf{SU}(3)/\mathsf{SU}(2)$ admits a unique invariant Einstein metric which coincides with the canonical metric. Consequently, the space $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$ admits a unique invariant Einstein metric.

Using general properties of symmetric spaces, we see that there are no invariant $\mathsf{Spin}(7)$-structures on the manifolds described in part b) of Theorem A (cf. Lemma 3.2). Combining this with a case-by-case analysis of the homogeneous spaces appearing in part a) gives the following result.

Theorem B. *The canonical presentations of all compact, simply connected, almost effective homogeneous spaces admitting an invariant $\mathsf{Spin}(7)$-structure are exhausted by $\frac{\mathsf{SU}(3)}{\{e\}}$, the infinite family $C_{k,\ell,m}$, for $k = \ell + m$, and the Calabi-Eckmann manifold $\frac{\mathsf{SU}(3)}{\mathsf{SU}(2)} \times \mathsf{SU}(2)$.*

It is remarkable that there are just a few examples of compact simply connected homogeneous spaces admitting invariant $\mathsf{Spin}(7)$-structures. This is different from the case of G_2-structures, where examples of this type are abundant (see [22, 31], and compare with the classification of compact almost effective homogeneous 7-manifolds given in [2]).

An example of invariant $\mathsf{Spin}(7)$-structure on the homogeneous space $\mathsf{SU}(3)/\{e\}$ was described in [12]. For the spaces $C_{\ell+m,\ell,m}$, with $\ell > m > 0$, and $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$, we describe the invariant Riemannian metrics and the invariant differential forms. This allows us to obtain a 5-parameter family of invariant $\mathsf{Spin}(7)$-structures of mixed type on both of them. In particular, for both spaces we show that there exists an invariant $\mathsf{Spin}(7)$-structure Φ inducing the normal metric and whose associated $\mathsf{Spin}(7)$-connection ∇ coincides with the canonical connection corresponding to the naturally reductive structure induced by g_Φ. From this, it follows that ∇ has parallel torsion.

The paper is organized as follows. In Section 2, we recall some basic facts on Spin(7)-structures. Homogeneous 8-manifolds with an invariant Spin(7)-structure are discussed in Section 3. In Section 4, we review the main properties of simply connected homogeneous spaces. The main theorems A and B are proved in Sections 5 and 6, respectively. In particular, the infinite family $C_{k,\ell,m}$ is studied in Section 6.1, the Calabi-Eckman manifold $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$ is described in Section 6.2 and $(\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2))/\Delta(\mathsf{SU}(2))) \times (\mathsf{SU}(2)/\mathsf{U}(1))$ is analysed in Section 6.3. Explicit examples of invariant Spin(7)-structures on $C_{\ell+m,\ell,m}$, with $\ell \geq m > 0$, and on the Calabi-Eckmann manifold are given is Section 7, where we also study the corresponding invariant Spin(7)-connection with torsion. Finally, in Appendix A we discuss the classification of all non-symmetric homogeneous presentations of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$.

We emphasize that the results of this paper are also useful to study compact 8-manifolds admitting other types of special structures, e.g. invariant PSU(3)-structures. This will be discussed in a forthcoming work.

2. Preliminaries

We begin recalling the main properties of 8-manifolds whose frame bundle admits a reduction to the Lie group $\mathsf{Spin}(7) \subset \mathsf{SO}(8)$. For more details, we refer the reader to [9, 15, 16, 21].

Consider the vector space $\mathbb{R}^8$, denote by $\{e_0, \ldots, e_7\}$ the canonical basis, and by $\{e^0, \ldots, e^7\}$ its dual basis. The group $\mathsf{Spin}(7)$ can be defined as the stabilizer in $\mathsf{GL}(8, \mathbb{R})$ of the following 4-form on $\mathbb{R}^8$:

$$\begin{aligned} \Phi_0 \quad = \quad & e^{0123} + e^{0145} - e^{0167} + e^{0246} + e^{0257} + e^{0347} - e^{0356} \\ & + e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} - e^{1247} + e^{1256}. \end{aligned} \tag{2.1}$$

$\mathsf{Spin}(7)$ is a compact, connected, simply connected Lie group of dimension 21. It is a subgroup of $\mathsf{SO}(8)$, as it preserves both the Euclidean inner product $g_0 = \sum_{i=1}^{8}(e^i)^2$ on $\mathbb{R}^8$ and the volume form $\frac{1}{14}\Phi_0 \wedge \Phi_0 = e^{01234567}$. Moreover, its center is $\mathbb{Z}_2 = \{\pm \mathrm{Id}_{\mathbb{R}^8}\}$ and $\mathsf{Spin}(7)/\mathbb{Z}_2 \cong \mathsf{SO}(7)$ (see [9, Thm. 4] for a proof).

A $\mathsf{Spin}(7)$-structure on a 8-dimensional manifold M is a reduction of the structure group of its frame bundle from $\mathsf{GL}(8, \mathbb{R})$ to $\mathsf{Spin}(7)$. As $\mathsf{Spin}(7)$ is the stabilizer of the 4-form Φ_0, such a reduction is characterized by the existence of a globally defined 4-form $\Phi \in \Omega^4(M)$ which can be pointwise identified with Φ_0 by means of an isomorphism $u : T_x M \to \mathbb{R}^8$. Any such form is called *admissible*, and it gives rise to a Riemannian metric g_Φ and to an orientation dV_Φ on M by the inclusion $\mathsf{Spin}(7) \subset \mathsf{SO}(8)$. We denote the Hodge operator associated with this metric and orientation by $\star$. Notice that Φ is self-dual, i.e., $\star\Phi = \Phi$. An explicit description of the metric g_Φ in terms of the 4-form Φ can be found, for instance, in [17, Sect. 4.3].

Remark 2.1. By dimension counting, the $\mathsf{GL}(8, \mathbb{R})$-orbit of Φ_0 is not open in $\Lambda^4((\mathbb{R}^8)^*)$. Consequently, an admissible 4-form Φ is not *stable* in the sense of Hitchin [14]. This differs significantly from the case of G_2-structures on 7-manifolds, which are defined by stable 3-forms satisfying a suitable positivity condition. In eight dimensions, stability occurs for 3- and 5-forms, and the corresponding geometric structures are related to the group $\mathsf{PSU}(3)$.

Since $\mathsf{Spin}(7)$ is both connected and simply connected, a connected 8-manifold M admitting a $\mathsf{Spin}(7)$-structure must be orientable and spin (with a preferred spin structure and orientation). These conditions are equivalent to the vanishing of the first and second Stiefel-Whitney classes of M. However, not every 8-dimensional Riemannian spin manifold admits $\mathsf{Spin}(7)$-structures. More precisely, this is a topological issue, which can be characterized as follows.

Proposition 2.2 ([21]). *An 8-dimensional orientable spin manifold M admits $\mathsf{Spin}(7)$-structures if and only if, for an appropriate choice of orientation, the following equation involving the Pontryagin*

classes $p_1(M)$, $p_2(M)$ and the Euler characteristic $\chi(M)$ of M holds

$$p_1^2(M) - 4p_2(M) + 8\chi(M) = 0$$

The intrinsic torsion of a $\mathsf{Spin}(7)$-structure can be identified with the covariant derivative of the defining 4-form Φ with respect to the Levi Civita connection ∇^{g_Φ} of g_Φ. When $\nabla^{g_\Phi}\Phi = 0$, the intrinsic torsion vanishes identically, the holonomy group of g_Φ is a subgroup of $\mathsf{Spin}(7)$ and g_Φ is Ricci-flat. In such a case, the $\mathsf{Spin}(7)$-structure is said to be *torsion-free* or *parallel*. By [12], the intrinsic torsion $\nabla^{g_\Phi}\Phi$ can be also identified with the 5-form $d\Phi$. Moreover, the $\mathsf{Spin}(7)$-module $\Lambda^5((\mathbb{R}^8)^*)$ splits into the direct sum of two irreducible submodules, say $\Lambda^5((\mathbb{R}^8)^*) \cong \mathcal{W}_1 \oplus \mathcal{W}_2$, allowing one to divide $\mathsf{Spin}(7)$-structures into four classes, which are completely characterized by $d\Phi$. Besides the class of parallel $\mathsf{Spin}(7)$-structures, corresponding to the condition $d\Phi = 0$, the following possibilities occur

- class $\mathcal{W}_1$: *balanced* $\mathsf{Spin}(7)$-structures, characterized by the condition $\star d\Phi \wedge \Phi = 0$;
- class $\mathcal{W}_2$: *locally conformal parallel* $\mathsf{Spin}(7)$-structures, characterized by the condition $d\Phi = \vartheta \wedge \Phi$;
- class $\mathcal{W}_1 \oplus \mathcal{W}_2$: $\mathsf{Spin}(7)$-structures of *mixed type*.

The 1-form ϑ is given by

$$\vartheta = -\frac{1}{7} \star (\star d\Phi \wedge \Phi) = \frac{1}{7} \star (\delta\Phi \wedge \Phi),$$

and it is called the *Lee form* of the $\mathsf{Spin}(7)$-structure. In particular, Φ is balanced if and only if $\vartheta = 0$, while $d\vartheta = 0$ whenever the $\mathsf{Spin}(7)$-structure is locally conformal parallel (l.c.p. for short), see e.g. [17, Lemma 4.5.2].

Finally, according to [15, Thm. 1.1], any 8-dimensional manifold M endowed with a $\mathsf{Spin}(7)$-structure Φ admits a unique metric connection ∇ with totally skew-symmetric torsion T, satisfying $\nabla\Phi = 0$. It is given by $\nabla := \nabla^{g_\Phi} + \frac{1}{2}T$, where $T = -\delta\Phi - \frac{7}{6} \star (\vartheta \wedge \Phi)$, and it is called the *characteristic connection* of (M, Φ).

3. Invariant $\mathsf{Spin}(7)$-structures on homogeneous spaces

We now focus on homogeneous spaces. We start with the following definition.

Definition 3.1. A $\mathsf{Spin}(7)$-structure Φ on an 8-dimensional manifold M is called *homogeneous* or *invariant* if there exists a connected Lie group G acting transitively and almost effectively on M preserving the 4-form Φ.

In this case, M is G-equivariantly diffeomorphic to the homogeneous space G/H, where H is the isotropy group of a fixed point $o \in M$, and Φ is a G-invariant 4-form on G/H with pointwise stabilizer isomorphic to $\mathsf{Spin}(7)$. Equivalently, the isotropy subgroup $\chi(H) \subset \mathsf{GL}(T_oM)$ is a subgroup of $\mathsf{Spin}(7)$, where $\chi : H \to \mathsf{GL}(T_oM)$ denotes the isotropy representation of $M = G/H$. Conversely, a homogeneous 8-manifold $M = G/H$ with $\chi(H) \subseteq \mathsf{Spin}(7)$ admits invariant $\mathsf{Spin}(7)$-structures.

As we are interested in compact examples, from now on we assume that G is compact. Then H is compact as well, and the Lie algebra $\mathfrak{g}$ of G admits a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. Moreover, we can identify $\mathfrak{m}$ with the tangent space T_oM, and the G-invariant 4-form Φ on G/H with an $\mathrm{Ad}(H)$-invariant 4-form on $\mathfrak{m}$, which we shall denote by the same letter. Since the G-action on $M = G/H$ is almost effective, the isotropy representation $\chi_* : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$ is injective, and we can identify the subalgebra $\mathfrak{h}$ with the isotropy subalgebra $\chi_*(\mathfrak{h})$ of the Lie algebra $\mathfrak{spin}(7) \subset \mathfrak{gl}(\mathfrak{m})$. Notice that the following constraints must hold

$$\dim(\mathfrak{g}) = \dim(\mathfrak{h}) + 8, \quad \mathrm{rk}\,\mathfrak{h} \leq 3.$$

The well-known interplay between G_2- and $\mathsf{Spin}(7)$-structures (see e.g. [10]) implies that every invariant G_2-structure on a compact homogeneous 7-manifold $N = L/K$ gives rise to an invariant

Spin_7-structure on $M = L/K \times \mathsf{U}(1)$ and, conversely, every invariant Spin_7-structure on the 8-manifold $M = L/K \times \mathsf{U}(1)$ induces an invariant G_2-structure on $N = L/K$. Consequently, in this case the complete list of homogeneous manifolds with an invariant $\mathsf{Spin}(7)$-structure can be obtained from the results of [22, 31].

In the next sections, we will deal with the classification of the canonical presentations of compact, simply connected, almost effective homogeneous 8-manifolds that admit invariant $\mathsf{Spin}(7)$-structures. The strategy to study this problem is the following. First, we consider all possible compact, simply connected almost effective, homogeneous 8-manifolds with their *canonical presentation*, and we determine those satisfying the characterization of Proposition 2.2. This gives the list of the spaces admitting $\mathsf{Spin}(7)$-structures. Then, for each space we investigate whether there exists an invariant admissible 4-form.

We conclude this section with some remarks.

Lemma 3.2. *A compact, simply connected, Riemannian symmetric space cannot admit any invariant $\mathsf{Spin}(7)$-structure.*

Proof. Since all invariant differential forms on a Riemannian symmetric space are closed (see e.g. [35, p. 250]), any invariant $\mathsf{Spin}(7)$-structure on a Riemannian symmetric space must be torsion-free. In particular, the corresponding invariant metric must be Ricci-flat. However, every compact simply connected Riemannian symmetric space is a direct product of irreducible symmetric spaces of compact type, which are Einstein with non-zero Einstein constant (cf. [6, 10.83]). □

More generally, since Ricci-flat homogeneous manifolds are flat [3], the class of compact connected homogeneous spaces admitting an invariant torsion-free $\mathsf{Spin}(7)$-structure is exhausted by flat tori.

4. Simply connected 8-dimensional homogeneous spaces

Let M be a compact, simply connected homogeneous space and let G' be a connected Lie group acting transitively and almost effectively on it. Starting from the corresponding presentation $M = G'/H'$ and using the results of [25, 28], it is always possible to obtain a presentation of the form $M = G/H$, where G is a compact, connected, simply connected, semisimple Lie group and $H \subset G$ is a connected closed subgroup (see e.g. [7, 8] for more details). The latter is known in the literature as the *canonical presentation* of the homogeneous space G'/H'. In the following, we restrict our attention to such presentations.

Proposition 4.1. *For a compact, simply connected, homogeneous space M with canonical presentation $M = G/H$, two possible cases occur:*

(I) $\operatorname{rk} N_G(H) = \operatorname{rk} H$;
(II) $\operatorname{rk} N_G(H) > \operatorname{rk} H$.

In the first case, $M = G/H$ is the direct product of indecomposable homogeneous spaces G_i/H_i which also satisfy (I), that is

$$M = G/H = G_1/H_1 \times \ldots G_k/H_k,$$

for some $k \geq 1$, with G_i compact connected, simply connected and semisimple and $H_i \subset G_i$ closed, for all $1 \leq i \leq k$. Such homogeneous spaces G_i/H_i are called prime.

In case (II), $M = G/H$ is the total space of a principal torus bundle over a product of prime homogeneous spaces. In particular, for any maximal torus T in a compact complement of H in $N_G(H)$, $M = G/H$ is the total space of the principal torus bundle

$$(H \cdot \mathrm{T})/H \to G/H \to G/(H \cdot \mathrm{T}),$$

where $H \cdot \mathrm{T}$ denotes $(H \times \mathrm{T})/H \cap \mathrm{T}$. Note that the base space $G/(H \cdot \mathrm{T})$ does not depend on the choice of T and $\operatorname{rk} N_G(H \cdot \mathrm{T}) = \operatorname{rk}(H \cdot \mathrm{T})$.

Example 4.2. Consider the simply connected coset $(\mathsf{G}_2 \times \mathsf{SU}(2))/(\mathsf{SU}(3) \times \mathsf{U}(1))$, where $\mathsf{SU}(3)$ is the maximal compact connected subgroup of G_2, and $\mathsf{U}(1) \subset \mathsf{SU}(2)$ is a maximal torus. It is easy to see that this space satisfies condition (I). Hence, it is the direct product of the prime homogeneous spaces $\mathbb{S}^6_{\mathrm{irr}} = \mathsf{G}_2 / \mathsf{SU}(3)$ and $\mathbb{CP}^1 = \mathsf{SU}(2)/\mathsf{U}(1)$.

Proposition 4.1 allows us to distinguish two classes of compact, simply connected homogeneous spaces. A particular example of (II) is the following (cf. [18, p. 80]).

Lemma 4.3 ([18]). *Let P be the total space of a T^{q-1}-principal bundle over $(\mathbb{S}^2)^{\times q}$, with $q \geq 2$. If P is simply connected, then it is diffeomorphic to $\mathbb{S}^2 \times (\mathbb{S}^3)^{\times (q-1)}$.*

Notice that for $q = 2$ one obtains the circle bundle

$$\mathbb{S}^1 \to M_{k,\ell} = (\mathsf{SU}(2) \times \mathsf{SU}(2))/\mathsf{U}(1)_{k,\ell} \to \mathbb{S}^2 \times \mathbb{S}^2.$$

The space $M_{k,\ell}$ is a compact simply connected 5-dimensional spin manifold with $H^2(M_{k,\ell}, \mathbb{Z}) = \mathbb{Z}$, and hence diffeomorphic to the product of spheres $\mathbb{S}^2 \times \mathbb{S}^3$. For $k = \ell = 1$, the space $M_{1,1}$ is diffeomorphic to $\mathsf{SO}(4)/\mathsf{SO}(2)$, and it can be viewed as the unit tangent bundle of $\mathbb{S}^3$, see also [27, p. 6358]. For $q = 3$, we get the space

$$C_{k,\ell,m} := (\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2))/\mathsf{U}(1)_{k,\ell,m},$$

where the embedding $\mathsf{U}(1)_{k,\ell,m} \subset \mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)$ will be specified later (see Section 6.1). When $\gcd(k, \ell, m) = 1$, the space $C_{k,\ell,m}$ is a torus bundle over $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$

$$\mathrm{T}^2 \to (\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2))/\mathsf{U}(1)_{k,\ell,m} \to \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2,$$

see Lemma 7.1 for a proof. Thus, as a manifold it is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$.

An inspection of the list of canonical presentations given in [8] (see also [4, Table 1]) combined with the results of [18, p. 81], allows us to obtain the canonical presentations for all compact, connected, simply connected, spin non-symmetric almost effective homogeneous 8-manifolds. They are described in Table 1.

More details on the cosets $(2) - (4)$ are given in Section 6, while further presentations of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$ are discussed in Appendix A. Notice that the spaces (3) and $(6) - (8)$ are all simply-connected homogeneous nearly Kähler manifolds.

5. Proof of Theorem A

In this section, we study the existence of $\mathsf{Spin}(7)$-structures on the spaces appearing in Table 1 using the topological characterization of Proposition 2.2. Since all of these manifolds are orientable and spin, we only need to examine the constraint

$$8\chi(M) = 4p_2(M) - p_1^2(M). \tag{5.1}$$

Recall that for a compact, connected, oriented 8-manifold M, the following identity holds (cf. [32])

$$\sigma(M) = \frac{1}{45}\langle 7p_2(M) - p_1^2(M), [M]\rangle, \tag{5.2}$$

where $\sigma(M)$ is the *signature* of M, namely the signature of the quadratic form associated to

$$Q : H^4(M, \mathbb{R}) \times H^4(M, \mathbb{R}) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \langle \alpha \cup \beta, [M]\rangle := \int_M \alpha \wedge \beta,$$

and $[M] \in H_8(M, \mathbb{Z})$ is the fundamental homology class defined by the orientation. Moreover, the $\hat{A}$-*genus* of M is given by

$$\hat{A}(M) = \frac{1}{5760}(7p_1^2(M) - 4p_2(M)). \tag{5.3}$$

	M^8	canonical presentation G/H
(1)	$\mathsf{SU}(3)$	$\dfrac{\mathsf{SU}(3)}{\{e\}}$
(2)	$\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$	$C_{k,\ell,m} := \dfrac{\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)}{\mathsf{U}(1)_{k,\ell,m}}, \quad k \geq \ell \geq m \geq 0, \; k > 0, \; \gcd(k,\ell,m) = 1$
(3)	$\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$	$\dfrac{\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)}{\Delta \mathsf{SU}(2)} \times \dfrac{\mathsf{SU}(2)}{\mathsf{U}(1)}$
(4)	$\mathbb{S}^5_{\mathsf{V} \oplus \mathbb{R}} \times \mathbb{S}^3$	$\dfrac{\mathsf{SU}(3)}{\mathsf{SU}(2)} \times \mathsf{SU}(2)$
(5)	$\mathsf{Sp}(2)$-full flag	$\dfrac{\mathsf{Sp}(2)}{\mathrm{T}^2_{\max}}$
(6)	$\mathbb{F}^3 \times \mathbb{S}^2$	$\dfrac{\mathsf{SU}(3)}{\mathrm{T}^2_{\max}} \times \dfrac{\mathsf{SU}(2)}{\mathsf{U}(1)}$
(7)	$\mathbb{CP}^3_{\mathfrak{m}_1 \oplus \mathfrak{m}_2} \times \mathbb{S}^2$	$\dfrac{\mathsf{Sp}(2)}{\mathsf{Sp}(1) \times \mathsf{U}(1)} \times \dfrac{\mathsf{SU}(2)}{\mathsf{U}(1)}$
(8)	$\mathbb{S}^6_{\mathrm{irr}} \times \mathbb{S}^2$	$\dfrac{\mathsf{G}_2}{\mathsf{SU}(3)} \times \dfrac{\mathsf{SU}(2)}{\mathsf{U}(1)}$

TABLE 1. Canonical presentations of compact simply connected spin almost effective non-symmetric homogeneous 8-manifolds.

Assume now that M^8 is also spin and let us denote by Σ its spinor bundle and by $D^g : \Gamma(\Sigma) \to \Gamma(\Sigma)$ the Dirac operator associated to a Riemannian metric g on it. Then, Σ decomposes as $\Sigma = \Sigma^+ \oplus \Sigma^-$, and one can consider the index of the (half) Dirac operator $D^g_+ : \Gamma(\Sigma^+) \to \Gamma(\Sigma^-)$, which is given by $\mathrm{ind}(D^g_+) := \dim \ker(D^g_+) - \dim \mathrm{coker}(D^g_+)$. By the Atiyah-Singer Index Theorem, $\mathrm{ind}(D^g_+)$ coincides with the $\hat{A}$-genus, i.e., $\mathrm{ind}(D^g_+) = \langle \hat{A}(M), [M] \rangle$. Moreover, if M admits a metric of positive scalar curvature, then $\hat{A}(M) = 0$.

Let us now prove part a) of Theorem A.

Proposition 5.1. *Among the manifolds described in Table 1, only those appearing in the first four rows admit $\mathsf{Spin}(7)$-structures.*

Proof. By [12, Sect. 7], we know that $\mathsf{SU}(3)$ admits $\mathsf{Spin}(7)$-structures. Let M be one of the manifolds $(2) - (4)$ of Table 1. As M is a product of spheres with at least one of odd-dimension, it is parallelizable by [19] (see also [29]). Consequently, it admits $\mathsf{Spin}(7)$-structures. Explicit examples of admissible 4-forms can be easily expressed in terms of a global coframe $\{e^0, \ldots, e^7\}$ providing the absolute parallelism.

We now prove that the remaining spaces of Table 1 do not satisfy the relation (5.1). Indeed, apart from the full flag manifold $\mathsf{Sp}(2)/\mathrm{T}^2_{\max}$, they are all products of the form $M = X^6 \times \mathbb{S}^2$, where X^6 is a 6-dimensional compact homogeneous nearly Kähler manifold. Therefore, it is easy to see that they satisfy $p_1^2(M) = p_2(M) = 0$, but their Euler characteristic is non-zero, since they are quotients of Lie groups of the same rank. For the same reason, the full flag manifold $M = \mathsf{Sp}(2)/\mathrm{T}^2_{\max}$ has $\chi(M) \neq 0$. Moreover, $\sigma(M) = \hat{A}(M) = 0$, and by (5.2) and (5.3) we deduce that $p_1^2(M) = p_2(M) = 0$. Thus, none of these manifolds satisfies (5.1). $\square$

We now prove part b) of Theorem A.

Proposition 5.2. *The 8-dimensional compact, simply connected, symmetric spaces admitting* $\mathsf{Spin}(7)$-*structures are exhausted by the Lie group* $\mathsf{SU}(3)$, *the product of spheres* $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$ *and* $\mathbb{S}^5 \times \mathbb{S}^3$, *the quaternionic projective space* $\mathbb{HP}^2$ *and the exceptional Wolf space* $\frac{\mathsf{G}_2}{\mathsf{SO}(4)}$.

Proof. Since an 8-manifold admitting $\mathsf{Spin}(7)$-structures is spin, we can focus on the list of compact simply connected spin symmetric spaces [11]. Up to a finite cover, we have to consider the following spaces

$$
\begin{aligned}
\mathsf{SU}(3) &= \frac{\mathsf{SU}(3) \times \mathsf{SU}(3)}{\Delta\,\mathsf{SU}(3)}, & \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2 &= \frac{\mathsf{SO}(4)}{\mathsf{SO}(3)} \times \frac{\mathsf{SO}(4)}{\mathsf{SO}(3)} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)}, \\
\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 &= \left(\frac{\mathsf{SU}(2)}{\mathsf{U}(1)}\right)^{\times 4}, & \mathbb{S}^4 \times \mathbb{S}^2 \times \mathbb{S}^2 &= \frac{\mathsf{SO}(5)}{\mathsf{SO}(4)} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)}, \\
\mathrm{Gr}_2(\mathbb{C}^4) &= \mathrm{Gr}_2^+(\mathbb{R}^6), & \mathbb{S}^5 \times \mathbb{S}^3 &= \frac{\mathsf{SO}(6)}{\mathsf{SO}(5)} \times \frac{\mathsf{SU}(2) \times \mathsf{SU}(2)}{\Delta\,\mathsf{SU}(2)}, \\
\mathbb{CP}^3 \times \mathbb{S}^2 &= \frac{\mathsf{Sp}(2)}{\mathsf{U}(2)} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)}, & \mathbb{S}^4 \times \mathbb{S}^4 &= \frac{\mathsf{SO}(5)}{\mathsf{SO}(4)} \times \frac{\mathsf{SO}(5)}{\mathsf{SO}(4)}, \\
\mathbb{HP}^2 &= \frac{\mathsf{Sp}(3)}{\mathsf{Sp}(2) \times \mathsf{Sp}(1)}, & \mathbb{S}^6 \times \mathbb{S}^2 &= \frac{\mathsf{SO}(7)}{\mathsf{SO}(6)} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)}, \\
\mathbb{S}^8 &= \frac{\mathsf{SO}(9)}{\mathsf{SO}(8)}, & & \frac{\mathsf{G}_2}{\mathsf{SO}(4)}.
\end{aligned}
$$

Among these spaces, only $\mathsf{SU}(3)$, $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^5 \times \mathbb{S}^3$, $\mathbb{HP}^2$, and $\frac{\mathsf{G}_2}{\mathsf{SO}(4)}$ satisfy (5.1). For the first three of them, we have $\chi = \sigma = \hat{A} = 0$. As for the quaternionic projective space and the exceptional Wolf space, they both satisfy $\chi = 3$, $\sigma = 1$ and $\hat{A} = 0$.   □

6. Proof of Theorem B

By Lemma 3.2, we know that all compact, simply connected, symmetric spaces cannot admit invariant $\mathsf{Spin}(7)$-structures. Moreover, an explicit example of an invariant $\mathsf{Spin}(7)$-structure on the homogeneous space $\mathsf{SU}(3)/\{e\}$ is constructed in [12, Sect. 7]. Thus, we only need to consider the remaining spaces appearing in part a) of Theorem A, namely

$$
C_{k,\ell,m} = \frac{\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)}{\mathsf{U}(1)_{k,\ell,m}} \quad (m \geq 0), \quad \frac{\mathsf{SU}(3)}{\mathsf{SU}(2)} \times \mathsf{SU}(2), \quad \frac{\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)}{\Delta\,\mathsf{SU}(2)} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)}.
$$

In order to simplify the presentation, we examine each case separately.

6.1. **The infinite family** $C_{k,\ell,m}$. Let $G := \mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)$ and

$$
H := \mathsf{U}(1)_{k,\ell,m} = \left\{ (z^k, z^\ell, z^m) : z \in \mathsf{U}(1) \right\}.
$$

Denote the Lie algebra of G by $\mathfrak{g} := 3\mathfrak{su}(2) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and let $\mathfrak{t}$ be the Lie algebra of a maximal torus of G. The elements of $\mathfrak{g}$ can be viewed as (6×6) complex block matrices of the form $\mathrm{diag}(X, Y, Z)$, with $X, Y, Z \in \mathfrak{su}(2)$. Up to conjugation, any 1-dimensional subalgebra inside $\mathfrak{g}$ is described by a homomorphism

$$
\rho_{k,\ell,m} : \mathfrak{u}(1) \to \mathfrak{g}, \quad ix \mapsto \mathrm{diag}\left(\begin{pmatrix} ikx & 0 \\ 0 & -ikx \end{pmatrix}, \begin{pmatrix} i\ell x & 0 \\ 0 & -i\ell x \end{pmatrix}, \begin{pmatrix} imx & 0 \\ 0 & -imx \end{pmatrix} \right), \quad k, \ell, m \in \mathbb{R}.
$$

The image of $\rho_{k,\ell,m}$ is the Lie algebra of a closed connected subgroup of G if and only if $k, \ell, m \in \mathbb{Q}$. Moreover, using the Weyl group and the outer automorphisms of G, it is always possible to reorder the elements of the triple (k, ℓ, m) in such a way that $k \geq \ell \geq m \geq 0$ and $k > 0$ and assume that

all k, ℓ, m are integers. The isotropy algebra $\mathfrak{h}$ is $\rho_{k,\ell,m}(\mathfrak{u}(1)) := \mathfrak{u}(1)_{k,\ell,m}$ and the Lie algebra $\mathfrak{t}$ is given by

$$\mathfrak{t} = \left\{ \mathrm{diag}\left(\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix}, \begin{pmatrix} iy & 0 \\ 0 & -iy \end{pmatrix}, \begin{pmatrix} iz & 0 \\ 0 & -iz \end{pmatrix} \right) : x, y, z \in \mathbb{R} \right\}.$$

Since the isotropy group H can be mapped by conjugation inside a maximal torus of G, any coset space of the form $(\mathsf{SU}(2))^{\times 3} / \mathsf{U}(1)$ is G-equivariantly diffeomorphic to $C_{k,\ell,m}$, where k, ℓ, m are integers such that $k \geq \ell \geq m \geq 0$ and $k > 0$. Moreover, since we are interested in the simply connected case, we can assume that the triple (k, ℓ, m) consists of co-prime integers.

Denote by $\langle \cdot, \cdot \rangle$ the bi-invariant metric on G defined as $\langle A, B \rangle = -2 \operatorname{tr}(AB)$ and let $\mathfrak{m} = \mathfrak{h}^\perp$ be the orthogonal complement of $\mathfrak{h}$ inside $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$. Then, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition. Thus, we can identify $\mathfrak{m}$ with the tangent space $T_o C_{k,\ell,m}$ of $C_{k,\ell,m}$ at the identity coset $o := eH$. Let us consider the following orthogonal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$:

$$
\begin{aligned}
e_1 &:= \mathrm{diag}(\sigma_1, 0, 0), & e_2 &:= \mathrm{diag}(\sigma_2, 0, 0), \\
e_3 &:= \mathrm{diag}(0, \sigma_1, 0), & e_4 &:= \mathrm{diag}(0, \sigma_2, 0), \\
e_5 &:= \mathrm{diag}(0, 0, \sigma_1), & e_6 &:= \mathrm{diag}(0, 0, \sigma_2), & (6.1)\\
e_7 &:= \tfrac{1}{c_7} \mathrm{diag}(mk\sigma_3, m\ell\sigma_3, -(k^2 + \ell^2)\sigma_3), & e_8 &:= \tfrac{1}{c_8} \mathrm{diag}(\ell\sigma_3, -k\sigma_3, 0), \\
e_9 &:= \mathrm{diag}(k\sigma_3, \ell\sigma_3, m\sigma_3),
\end{aligned}
$$

where

$$\sigma_1 := \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 := \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 := \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$c_7 := \sqrt{k^2 + \ell^2}\sqrt{k^2 + \ell^2 + m^2}$, and $c_8 := \sqrt{k^2 + \ell^2}$. Notice that $\langle e_i, e_i \rangle = 1$, for all $i = 1, \ldots, 8$, and that $\langle e_9, e_9 \rangle = k^2 + \ell^2 + m^2$. Moreover, $\mathfrak{h} = \mathrm{span}_\mathbb{R}\{e_9\}$ and $\mathfrak{m} = \mathrm{span}_\mathbb{R}\{e_1, \ldots, e_8\} \cong \mathbb{R}^8$.

Proposition 6.1. *The space $C_{k,\ell,m} = G/H$, with $k \geq \ell \geq m \geq 0$, $k > 0$ and $\gcd(k, \ell, m) = 1$, admits G-invariant $\mathsf{Spin}(7)$-structures if and only if $k - \ell - m = 0$.*

Proof. Let us denote by $\{E_{ij} : 1 \leq i < j \leq 8\}$ the basis of $\mathfrak{so}(8)$ given by the skew-symmetric matrices E_{ij} with -1 in the (i, j)-entry, 1 in the (j, i)-entry and zeroes elsewhere. The orthogonal transformation $\chi_*(e_9)|_\mathfrak{m} \in \mathfrak{so}(\mathfrak{m})$ is given by $\chi_*(e_9)|_\mathfrak{m} = -kE_{12} - \ell E_{34} - mE_{56}$, since

$$
\begin{aligned}
\mathrm{ad}(e_9)e_1 &= -ke_2, & \mathrm{ad}(e_9)e_2 &= ke_1, & \mathrm{ad}(e_9)e_3 &= -\ell e_4, & \mathrm{ad}(e_9)e_4 &= \ell e_3, \\
\mathrm{ad}(e_9)e_5 &= -me_6, & \mathrm{ad}(e_9)e_6 &= me_5, & \mathrm{ad}(e_9)e_7 &= 0, & \mathrm{ad}(e_9)e_8 &= 0.
\end{aligned} \quad (6.2)
$$

Thus, the isotropy action of $\mathfrak{h}$ on $\mathfrak{m}$ yields the following subalgebra of $\mathfrak{so}(8) = \mathfrak{so}(\mathfrak{m})$:

$$\chi_*(\mathfrak{h}) = \mathrm{ad}(\mathfrak{h})|_\mathfrak{m} = \left\{ \begin{pmatrix} \begin{array}{cc} 0 & kx \\ -kx & 0 \end{array} & & & \\ & \begin{array}{cc} 0 & \ell x \\ -\ell x & 0 \end{array} & & \\ & & \begin{array}{cc} 0 & mx \\ -mx & 0 \end{array} & \\ & & & \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \end{pmatrix} : x \in \mathbb{R} \right\}.$$

By using the basis of $\mathfrak{spin}_7 \subset \mathfrak{so}_8$ given in terms of the skew-symmetric matrices E_{ij} (cf. e.g. [5]), we see that a Cartan subalgebra of $\mathfrak{spin}(7)$ which occurs as the lift of a Cartan subalgebra of $\mathfrak{so}(7)$

has the following expression:

$$\mathfrak{t}^3 = \left\{ \left(\begin{array}{cccc} \begin{array}{cc} 0 & x \\ -x & 0 \end{array} & & & \\ & \begin{array}{cc} 0 & y \\ -y & 0 \end{array} & & \\ & & \begin{array}{cc} 0 & z \\ -z & 0 \end{array} & \\ & & & \begin{array}{cc} 0 & -(x-y-z) \\ x-y-z & 0 \end{array} \end{array} \right) : x, y, z \in \mathbb{R} \right\} \subset \mathfrak{spin}(7).$$

By comparing $\chi_*(\mathfrak{h})$ with $\mathfrak{t}^3$, we see that the action of $\mathfrak{h}$ on $\mathfrak{m}$ is equivalent to the action of $\mathfrak{t}^3$ on $\mathbb{O} \cong \mathbb{R}^8$ if and only if $k - \ell - m = 0$. Consequently, $C_{k,\ell,m}$ admits invariant $\mathsf{Spin}(7)$-structures if and only if $k = \ell + m$. □

Explicit examples of invariant $\mathsf{Spin}(7)$-structures on $C_{\ell+m,\ell,m}$ will be given in Section 7.1. In the next two examples, we describe some special spaces belonging to the family $C_{k,\ell,m}$.

Example 6.2. For $k = \ell = 1$ and $m = 0$, the space $C_{1,1,0}$ coincides with the direct product $(\mathsf{SO}(4)/\mathsf{SO}(2)) \times \mathsf{SU}(2) = \mathbb{V}_{4,2} \times \mathbb{S}^3$, where we recall that $\mathbb{V}_{4,2} \cong \mathbb{S}^3 \times \mathbb{S}^2$ (cf. [27]). Here, the 1-dimensional Lie subalgebra $\mathfrak{u}(1)_{1,1} \subset \mathfrak{t}^2$, where $\mathfrak{t}^2$ is a maximal torus of $\mathsf{SO}(4)$, corresponds to the diagonal embedding of $\mathfrak{u}(1)$ in $\mathfrak{so}(4)$, and the existence of an invariant $\mathsf{Spin}(7)$-structure follows from the inclusions $\mathfrak{u}(1)_{1,1} \subset \mathfrak{t}^2 \subset \mathfrak{t}^3$.

Example 6.3. Consider the direct product of the group manifold $\mathsf{SU}(2) \times \mathsf{SU}(2) = \mathsf{Spin}(4)$ with the homogeneous space $\mathbb{S}^2 = \mathsf{SU}(2)/\mathsf{U}(1)$,

$$M = G/H = G'/H' \times G''/H'' = \frac{\mathsf{SU}(2) \times \mathsf{SU}(2)}{\{e\}} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)}.$$

The isotropy algebra is given by $\mathfrak{h} := \{0\} \oplus \mathfrak{u}(1) \cong \mathfrak{u}(1)$, and the corresponding isotropy action is effective. In particular, $\chi_*(\mathfrak{u}(1))$ acts trivially on the tangent space of $G'/H' = \mathsf{SU}(2) \times \mathsf{SU}(2)$, while $\mathsf{U}(1)$ sits diagonally inside $\mathsf{SU}(2)$ it induces an irreducible representation when restricted to $T_{eH''}G''/H''$. Thus, this manifold belongs to the family $C_{k,\ell,m}$ for $k = 1$ and $\ell = m = 0$, and we have the obvious diffeomorphisms

$$\frac{\mathsf{SU}(2) \times \mathsf{SU}(2)}{\{e\}} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)} = C_{1,0,0} \cong C_{1,0,0} \cong C_{0,1,0}.$$

The reductive decomposition of $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is

$$\mathfrak{g} \cong \mathfrak{so}(4) \oplus \mathfrak{su}(2) = \mathfrak{u}(1) \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{n} \oplus V^2 = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathsf{U},$$

where $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 = \mathfrak{n} \cong T_e \mathsf{Spin}(4)$, and $\mathsf{U} := V^2 = [\mathbb{C}]_\mathbb{R}$ denotes the realification of the standard representation of $\mathsf{U}(1)$ on $\mathbb{C}$. Since the triple $(1,0,0)$ does not satisfy the condition $k = \ell + m$, the space $M = G'/H' \times G''/H''$ cannot admit any invariant $\mathsf{Spin}(7)$-structure by Proposition 6.1.

6.2. The Calabi-Eckmann manifold $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$.

This space is the direct product of the homogeneous spaces $\mathsf{SU}(3)/\mathsf{SU}(2)$ and $\mathsf{SU}(2)$. The former is the canonical presentation of the unit 5-sphere $\mathbb{S}^5 \subset \mathbb{C}^3$ acted on transitively and almost effectively by the Lie group $\mathsf{SU}(3)$ with isotropy group at $(1,0,0) \in \mathbb{C}^3$ given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \mathsf{SU}(3) : A \in \mathsf{SU}(2) \right\} \cong \mathsf{SU}(2).$$

Since the isotropy group $\mathsf{SU}(2)$ lies inside $\mathsf{SU}(3) \subset \mathsf{G}_2 \subset \mathsf{Spin}(7)$, and since G_2 is a maximal subgroup of $\mathsf{Spin}(7)$, the homogeneous space $M = (\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$ admits invariant $\mathsf{Spin}(7)$-structures. Explicit examples will be discussed in Section 7.2.

Remark 6.4. The space $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$ is a *Calabi-Eckmann manifold*, i.e., a complex homogeneous non-Kähler manifold diffeomorphic to the product of two odd-dimensional spheres of dimension greater than two. In particular, it is a torus bundle over $\mathbb{CP}^2 \times \mathbb{CP}^1$,

$$\mathrm{T}^2 \cong \frac{\mathsf{U}(2) \times \mathsf{U}(1)}{\mathsf{SU}(2)} \longrightarrow \frac{\mathsf{SU}(3) \times \mathsf{SU}(2)}{\mathsf{SU}(2) \times \{e\}} \longrightarrow \frac{\mathsf{SU}(3) \times \mathsf{SU}(2)}{\mathsf{U}(2) \times \mathsf{U}(1)},$$

and, consequently, a C-space with $H^2(M, \mathbb{Z}) = 0$ (see [18]).

6.3. **The space** $(\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2))/\Delta(\mathsf{SU}(2))) \times (\mathsf{SU}(2)/\mathsf{U}(1))$**.** For the sake of convenience, from now on we let

$$L^6 := \frac{\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)}{\Delta(\mathsf{SU}(2))}, \quad X^6 := \frac{\mathsf{SO}(3) \times \mathsf{SO}(3) \times \mathsf{SO}(3)}{\Delta\,\mathsf{SO}(3)}.$$

Both manifolds L^6 and X^6 are *Ledger-Obata spaces*, i.e., of the form $(K \times K \times K)/\Delta K$, with K a compact simple Lie group and $\Delta K = \{(k, k, k) : k \in K\}$. Moreover, there is a natural isomorphism between the compact homogeneous space $(K \times K \times K)/\Delta K$ and the compact semisimple Lie group $K \times K$. Consequently, the corresponding 8-manifolds are diffeomorphic to $C_{1,0,0} = \mathsf{Spin}(4) \times \mathbb{S}^2$. Notice however that $L^6 \times (\mathsf{SU}(2)/\mathsf{U}(1))$ does not belong to the family $C_{k,\ell,m}$.

Since $\mathsf{SU}(2)/\mathbb{Z}_2 \cong \mathsf{SO}(3)$, the effective coset $X^6 \times (\mathsf{SO}(3)/\mathsf{SO}(2))$ is covered by the almost effective simply connected coset $L^6 \times (\mathsf{SU}(2)/\mathsf{U}(1))$. Thus, to conclude the proof of Theorem B, it is sufficient to show that the space $X^6 \times (\mathsf{SO}(3)/\mathsf{SO}(2))$ does not admit any invariant $\mathsf{Spin}(7)$-structure. To this aim, we will first describe the isotropy representation, and then the space of invariant forms.

Let $\tilde{\mathfrak{k}} = \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k}$ and $\Delta\mathfrak{k} = \{(X, X, X) : X \in \mathfrak{k}\}$ be the Lie algebras of $K \times K \times K$ and ΔK, respectively. A natural choice of an $\mathrm{Ad}(\Delta K)$-invariant complement of $\Delta\mathfrak{k}$ in $\tilde{\mathfrak{k}}$ is given for instance by (see e.g. [26])

$$\mathfrak{n} = \left\{ (a_1 X, a_2 X, a_3 X) \in \tilde{\mathfrak{k}} \; : \; X \in \mathfrak{k}, \; a_i \in \mathbb{R}, \; \sum_{i=1}^{3} a_i = 0 \right\},$$

and then $\Delta\mathfrak{k} = \left\{ (aX, aX, aX) \in \tilde{\mathfrak{k}} \; : \; X \in \mathfrak{k}, \; a \in \mathbb{R} \right\}$. In our case, $\mathfrak{k} = \mathfrak{so}(3) \cong \mathfrak{su}(2)$.

Consider on $\tilde{\mathfrak{k}}$ the bi-invariant metric $\langle A, B \rangle = -(1/2)\mathrm{tr}(AB)$, which is a multiple of the corresponding Killing form. The Lie algebra $\mathfrak{so}(3)$ can be identified with the span of $\{E_{12}, E_{13}, E_{23}\}$, and the matrices

$$h_1 := E_{12} + E_{45} + E_{78}, \quad h_2 := E_{13} + E_{46} + E_{79}, \quad h_3 := E_{14} + E_{56} + E_{89}$$

generate $\Delta\mathfrak{so}(3) \cong \mathfrak{so}(3)$. By using the Gram-Schmidt process, we see that an orthogonal splitting of $\mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle$ is given by

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 = \{a_1(X, 0, -X) \; : \; X \in \mathfrak{k}, \; a_1 \in \mathbb{R}\} \oplus \left\{ a_2 \left(-\frac{1}{2}X, X, -\frac{1}{2}X \right) \; : \; X \in \mathfrak{k}, \; a_2 \in \mathbb{R} \right\},$$

where both $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are irreducible. Therefore

$$e_1 = \tfrac{1}{\sqrt{2}}(E_{12} - E_{78}), \qquad e_2 = \tfrac{1}{\sqrt{2}}(E_{13} - E_{79}), \qquad e_3 = \tfrac{1}{\sqrt{2}}(E_{23} - E_{89})$$

$$e_4 = -\tfrac{\sqrt{6}}{6}(E_{12} - 2E_{45} - E_{78}), \quad e_5 = -\tfrac{\sqrt{6}}{6}(E_{13} - 2E_{46} - E_{79}), \quad e_6 = -\tfrac{\sqrt{6}}{6}(E_{23} - 2E_{56} - E_{89})$$

form a $\langle \cdot , \cdot \rangle$-orthonormal basis of $\mathfrak{n}$ such that $\mathfrak{n}_1 = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3\}$ and $\mathfrak{n}_2 = \mathrm{span}_{\mathbb{R}}\{e_4, e_5, e_6\}$, respectively. Let us denote by E the standard representation of $\mathfrak{so}(3)$. Then, one may identify $\mathfrak{n}_1 = \mathsf{E}$ and $\mathfrak{n}_2 = \mathsf{E}'$, where E' denotes another copy of E.

Set now $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k} = 3\mathfrak{k} \oplus \mathfrak{k} = 4\mathfrak{so}(3) = 4\mathfrak{su}(2)$, and consider the Lie algebra

$$\mathfrak{h} := \Delta\mathfrak{so}(3) \oplus \mathfrak{u}(1) .$$

This is a subalgebra of $\mathfrak{so}(8)$, since $C_{\mathsf{SO}(8)}(\Delta\mathfrak{so}(3)) = \mathsf{U}(1) \times \mathsf{U}(1)$. Its defining representation is $\mathsf{E} \oplus \mathsf{E}' \oplus \mathsf{U}$, where U denotes the standard representation of $\mathfrak{u}(1)$ on $\mathbb{C}$. Since $\mathfrak{u}(1)$ sits inside the last summand of $\mathfrak{g}$, $\mathfrak{h}$ sits inside $\mathfrak{g}$ and the pair $(\mathfrak{g}, \mathfrak{h})$ induces the 8-dimensional effective homogeneous space $X^6 \times \frac{\mathsf{SO}(3)}{\mathsf{SO}(2)}$. Summing up, a reductive decomposition of $\mathfrak{g} = 4\mathfrak{so}_3$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{n} \oplus \mathsf{U} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathsf{U} = \mathsf{E} \oplus \mathsf{E}' \oplus \mathsf{U},$$

where $\mathfrak{n}$ coincides with the tangent space to X^6 and U with the tangent space to $\frac{\mathsf{SU}(2)}{\mathsf{U}(1)}$ at the identity coset. Moreover, an orthonormal basis of $\mathfrak{m}$ is given by

$$\{e_1, \ldots, e_6, e_7 := -E_{10,11}, \ e_8 := -E_{10,12}\},$$

so that $\mathsf{U} = \mathrm{span}_{\mathbb{R}}\{e_7, e_8\}$. The elements $\{h_1, h_2, h_3\}$ defined above, together with $h_4 := E_{11,12}$ generate the isotropy algebra $\mathfrak{h}$, which acts on $\mathfrak{m}$ via the isotropy representation as follows

$$\begin{array}{lll}
\chi_*(h_1)e_1 = 0, & \chi_*(h_2)e_1 = -e_3, & \chi_*(h_3)e_1 = e_2, \\
\chi_*(h_1)e_2 = e_3, & \chi_*(h_2)e_2 = 0, & \chi_*(h_3)e_2 = -e_1, \\
\chi_*(h_1)e_3 = -e_2, & \chi_*(h_2)e_3 = e_1, & \chi_*(h_3)e_3 = 0, \\
\chi_*(h_1)e_4 = 0, & \chi_*(h_2)e_4 = -e_6, & \chi_*(h_3)e_4 = e_5, \\
\chi_*(h_1)e_5 = e_6, & \chi_*(h_2)e_5 = 0, & \chi_*(h_3)e_5 = -e_4, \\
\chi_*(h_1)e_6 = -e_5, & \chi_*(h_2)e_6 = e_4, & \chi_*(h_3)e_6 = 0,
\end{array} \tag{6.3}$$

and

$$\chi_*(h_4)e_7 = -e_8, \quad \chi_*(h_4)e_8 = e_7. \tag{6.4}$$

We can now determine the invariant forms.

Lemma 6.5. *Let* $X^6 \times (\mathsf{SO}(3)/\mathsf{SO}(2)) = G/H$. *Then, the following hold:*

1) *the space of G-invariant 1-forms is trivial;*
2) *the space of G-invariant 2-forms is 2-dimensional and it is generated by $\omega_1 := e^{14} + e^{25} + e^{36}$ and $\omega_2 := e^{78}$;*
3) *the space of G-invariant 3-forms is 2-dimensional and it is generated by $\{e^{123}, e^{456}\}$;*
4) *the space of G-invariant 4-forms 2-dimensional and it is generated by $\{\omega_1 \wedge \omega_1, \omega_1 \wedge \omega_2\}$.*

Proof. The assertion for invariant 1-forms immediately follows from (6.3) and (6.4). For invariant 2-forms on X^6, we have the equivariant decomposition

$$(\Lambda^2 \mathfrak{n}^*)^H = (\Lambda^2 \mathfrak{n}_1^*)^H \oplus (\mathfrak{n}_1^* \wedge \mathfrak{n}_2^*)^H \oplus (\Lambda^2 \mathfrak{n}_2^*)^H .$$

Here, the first and the third module vanish, while from (6.3) we see that the second module is generated by ω_1. For the invariant 3-forms on X^6, we see that only the modules $(\Lambda^3 \mathfrak{n}_1^*)^H$ and $(\Lambda^3 \mathfrak{n}_2^*)^H$ are non-trivial. In particular, the 9-dimensional spaces $\Lambda^2 \mathfrak{n}_1^* \wedge \mathfrak{n}_2^*$ and $\mathfrak{n}_1^* \wedge \Lambda^2 \mathfrak{n}_2^*$ do not contain any invariant element. Now, the claim for G/H follows from the orthogonal decompositions

$$\begin{aligned}
(\Lambda^3 \mathfrak{m}^*)^H &= (\Lambda^3 \mathfrak{n}^*)^H \oplus (\Lambda^2 \mathfrak{n}^* \wedge \mathsf{U}^*)^H \oplus (\mathfrak{n}^* \wedge \Lambda^2 \mathsf{U}^*)^H, \\
(\Lambda^4 \mathfrak{m}^*)^H &= (\Lambda^4 \mathfrak{n}^*)^H \oplus (\Lambda^3 \mathfrak{n}^* \wedge \mathsf{U}^*)^H \oplus (\Lambda^2 \mathfrak{n}^* \wedge \Lambda^2 \mathsf{U}^*)^H.
\end{aligned}$$

$\square$

Proposition 6.6. *The space* $X^6 \times (\mathsf{SO}(3)/\mathsf{SO}(2))$ *cannot admit any invariant* $\mathsf{Spin}(7)$-*structure.*

Proof. We show that there are no invariant admissible 4-forms on this space. Let us consider the generic $\mathrm{Ad}(H)$-invariant 4-form on $\mathfrak{m}$

$$\Phi := a_1(e^{1245} + e^{1346} + e^{2356}) + a_2(e^{1478} + e^{2578} + e^{3678}),$$

where $a_1, a_2 \in \mathbb{R}$. By [17, Thm. 4.3.3], if Φ defines a Spin(7)-structure inducing the orientation $e^{12345678}$, then the norm of a vector $u = \sum_{i=1}^{8} u^k e_k \in \mathfrak{m}$ with $u^1 \neq 0$ is proportional to the determinant of the 7×7 matrix (a_{ij}) with the following entries

$$a_{ij} := (e_i \lrcorner u \lrcorner \Phi) \wedge (e_j \lrcorner u \lrcorner \Phi) \wedge (u \lrcorner \Phi)(e_2, \dots, e_8), \quad 2 \leq i \leq j \leq 8.$$

Considering $u = e_1$, an easy computation shows that all a_{ij} vanish. Thus, Φ cannot define an invariant Spin(7)-structure. $\qquad\square$

7. Explicit examples of invariant Spin(7)-structures

7.1. **The infinite family $C_{k,\ell,m}$.** By Proposition 6.1, the space $C_{k,\ell,m}$ admits invariant Spin(7)-structures if and only if $k = \ell + m$. In particular, since $\ell \geq m \geq 0$ with $\ell > 0$, two different cases arise, namely

$$C_{\ell+m,\ell,m}, \text{ with } \ell > m > 0, \qquad C_{\ell,\ell,0}, \text{ with } \ell > 0.$$

For these two classes of homogeneous manifolds, the invariant objects (metrics, forms) are different. Here, we examine the family $C_{k,\ell,m} = G/H$ for $k > \ell > m > 0$, where $G = \mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)$ and $H = \mathsf{U}(1)_{k,\ell,m}$. In our computations, we shall keep on using the notation introduced in Section 6.1.

Let $\mathfrak{m}_0 := \mathfrak{h}^\perp$ denote the orthogonal complement of $\mathfrak{h}$ inside the maximal torus $\mathfrak{t}$ with respect to $\langle \cdot, \cdot \rangle$. The space $\mathfrak{m}_0$ is spanned by $\{e_7, e_8\}$ and it is an abelian Lie algebra, i.e., $[e_7, e_8] = 0$. Thus, whenever $k, \ell, m \in \mathbb{Q}$, it generates a closed connected 2-dimensional abelian subgroup of T^3, i.e., a 2-torus, which we denote by $\mathrm{T}_{k,\ell,m}^2 \subset \mathrm{T}^3$. This induces the 7-dimensional homogeneous space (cf. [27, 31])

$$Q_{k,\ell,m} = (\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2))/\mathrm{T}_{k,\ell,m}^2 \cong \mathbb{S}^3 \times \mathbb{S}^2 \times \mathbb{S}^2,$$

which is a circle bundle $\mathsf{q} : Q_{k,\ell,m} \to \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ over $N^6 := \mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)/\mathrm{T}^3 = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$.

Moving to the family $C_{k,\ell,m}$, the reductive decomposition described before now reads

$$\mathfrak{g} = \mathfrak{u}(1)_{k,\ell,m} \oplus \mathfrak{m}, \quad \mathfrak{m} \cong T_o C_{k,\ell,m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_0 = \mathfrak{p} \oplus \mathfrak{m}_0 \tag{7.1}$$

where $\mathfrak{m}_1 = \mathrm{span}_{\mathbb{R}}\{e_1, e_2\}$, $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{e_3, e_4\}$, $\mathfrak{m}_3 = \mathrm{span}_{\mathbb{R}}\{e_5, e_6\}$ and $\mathfrak{p} := \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ coincides with the 6-dimensional tangent space of $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. In the following, every $\mathfrak{m}_i$, $i = 0, 1, 2, 3$, will be viewed as an $\mathrm{Ad}(H)$-module.

Remark 7.1. When $\gcd(k, \ell, m) = 1$, the simply connected coset $C_{k,\ell,m}$ is a non-Kähler C-space in the sense of Wang (cf. [1, 30]), i.e., a simply connected compact homogeneous complex manifold which is not Kähler. Indeed, since the integers k, ℓ, m are assumed to be relatively prime, the action of H on G

$$(e^{i\phi}, (x, y, z)) \mapsto (e^{ik\phi}x, e^{i\ell\phi}y, e^{im\phi}z),$$

is free. Consequently, the projection $\mathsf{c} : C_{k,\ell,m} \to N^6 = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ defines a principal torus bundle,

$$\mathrm{T}_{k,\ell,m}^2 \cong \mathrm{T}^3/H \longrightarrow C_{k,\ell,m} = G/H \xrightarrow{\ \mathsf{c}\ } N^6 = G/\mathrm{T}^3 = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2,$$

and we have the following diagram:

$$
\begin{array}{ccccc}
\mathsf{U}(1)_{k,\ell,m} & \lhook\joinrel\longrightarrow & \mathrm{T}^3 & \longleftarrow & \mathrm{T}^2_{k,\ell,m} \\
& & \downarrow & & \downarrow \\
Q_{k,\ell,m} \cong \mathbb{S}^3 \times \mathbb{S}^2 \times \mathbb{S}^2 & \longleftarrow & G = (\mathsf{SU}(2))^{\times 3} & \longrightarrow & C_{k,\ell,m} \cong \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2 \\
& {\scriptstyle\mathsf{q}}\searrow & \downarrow {\scriptstyle\pi} & {\scriptstyle\mathsf{c}}\swarrow & \\
& & N^6 = G/\mathrm{T}^3 \cong \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 & &
\end{array}
$$

Since $C_{k,\ell,m}$ is a $\mathrm{T}^2_{k,\ell,m}$-bundle over the (full) flag manifold $N^6 = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, it must be a C-space. Any complex structure J_0 on $\mathfrak{m}_0$ is automatically $\mathrm{Ad}(H)$-invariant. Hence, whenever $J_\mathfrak{p}$ defines an $\mathrm{Ad}(\mathrm{T}^3)$-invariant complex structure on $N^6 = G/\mathrm{T}^3$, the endomorphism $J_\mathfrak{m} := J_\mathfrak{p} + J_{\mathfrak{m}_0}$ defines an $\mathrm{Ad}(H)$-invariant complex structure on $C_{k,\ell,m}$.

In order to describe the set of G-invariant metrics on $C_{k,\ell,m}$, or, equivalently, the space of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$, we first need to examine the properties of the modules $\mathfrak{m}_i, i = 0, \ldots, 3$.

Lemma 7.2. *Assume that $k > \ell > m > 0$. Then, the $\mathfrak{h}$-modules $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$ are pairwise inequivalent and irreducible. In contrast, $\mathfrak{m}_0$ decomposes into two irreducible 1-dimensional sub-modules, which we denote by $\mathfrak{m}_4 := \mathrm{span}_\mathbb{R}\{e_7\}$ and $\mathfrak{m}_5 := \mathrm{span}_\mathbb{R}\{e_8\}$, respectively. These submodules are mutually equivalent, and they are not equivalent to $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$. Consequently, the orthogonal $\mathrm{Ad}(H)$-invariant decomposition*

$$
\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_0 = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5
$$

is not in general unique.

Proof. By (6.2), we see that the weights of the adjoint action of H on $\mathfrak{m} \cong \mathbb{R}^8$ are

$$
\left(\exp\left(\pm 2k\sqrt{-1}\varphi \right), \ \exp\left(\pm 2\ell\sqrt{-1}\varphi \right), \ \exp\left(\pm 2m\sqrt{-1}\varphi \right), \ 1, \ 1 \right), \quad \varphi \in [0, 2\pi].
$$

Since the isotropy group H acts with different weights on $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$, whenever $k \neq \ell \neq m \neq 0$, it follows that $\mathfrak{m}_1, \mathfrak{m}_2$, and $\mathfrak{m}_3$ are mutually inequivalent. Now, although $\mathfrak{m}_0$ is irreducible under the adjoint action of $\mathrm{T}^2_{k,\ell,m}$, it decomposes into two equivalent $\mathrm{Ad}(H)$-invariant submodules, which are generated by e_7 and e_8, respectively. Moreover, since $[\mathfrak{h}, \mathfrak{m}_0] = 0$, we see that $\mathrm{Ad}(H)|_{\mathfrak{m}_0} = \mathrm{Id}$. Hence, $\mathfrak{m}_4$ and $\mathfrak{m}_5$ can be replaced by any pair of orthogonal 1-dimensional submodules in $\mathfrak{m}_0$. $\square$

Fix some angle $\theta \in [0, 2\pi]$, consider the vectors

$$
e_7^\theta := \cos(\theta)\, e_7 + \sin(\theta)\, e_8, \quad e_8^\theta := -\sin(\theta)\, e_7 + \cos(\theta)\, e_8,
$$

and let $\mathfrak{m}_4^\theta := \mathrm{span}_\mathbb{R}\{e_7^\theta\}$, $\mathfrak{m}_5^\theta := \mathrm{span}_\mathbb{R}\{e_8^\theta\}$. Then, the H-module $\mathfrak{m}_0^\theta := \mathfrak{m}_4^\theta \oplus \mathfrak{m}_5^\theta$ is equivalent to $\mathfrak{m}_0 = \mathfrak{m}_4 \oplus \mathfrak{m}_5$, and the pair $\{e_7^\theta, e_8^\theta\}$ is an orthonormal basis of $(\mathfrak{m}_0^\theta, \langle \cdot, \cdot \rangle)$. Now, as a consequence of Lemma 7.2, we obtain the following.

Proposition 7.3. *Assume that $k > \ell > m > 0$. Let Q be the normal metric on $C_{k,\ell,m}$ induced by the bi-invariant metric $\langle A, B \rangle = -2\,\mathrm{tr}(AB)$ on $\mathfrak{g}$. Then, up to scaling, any G-invariant metric on $C_{k,\ell,m}$ is given by*

$$
(\ ,\ )_{y_1, y_2, y_3, y_4, y_5, \theta} := y_1^2\, Q|_{\mathfrak{m}_1} + y_2^2\, Q|_{\mathfrak{m}_2} + y_3^2\, Q|_{\mathfrak{m}_3} + y_4^2\, Q|_{\mathfrak{m}_4^\theta} + y_5^2\, Q|_{\mathfrak{m}_5^\theta},
$$

for some positive real numbers $y_1, \ldots, y_5$. Thus, the space of G-invariant metrics on $C_{k,\ell,m}$, with $k > \ell > m > 0$, is 6-dimensional.

Let us now examine the spaces of invariant differential forms on $C_{k,\ell,m}$. By the general theory of C-spaces described in [1], we can construct an equivariant isomorphism

$$\phi : \mathfrak{u}(1)_{k,\ell,m} = Z_{\mathfrak{g}}(\mathfrak{u}(1)_{k,\ell,m}) \to H^2(M,\mathbb{R}),$$

from which we deduce that $H^2(C_{k,\ell,m}, \mathbb{Z}) \cong \mathbb{Z}$. This also follows from the exact sequence (see [1, 18])

$$H^1(\mathrm{T}^2, \mathbb{Z}) \cong \mathbb{Z}^2 \longrightarrow H^2(N^6, \mathbb{Z}) \cong \mathbb{Z}^3 \longrightarrow H^2(M, \mathbb{Z}).$$

In more geometric terms, any element $\xi \in \mathfrak{m}_0 + C_{\mathfrak{p}}(\mathfrak{h})$, where $C_{\mathfrak{p}}(\mathfrak{h})$ denotes the centralizer of the isotropy subalgebra $\mathfrak{h}$ in $\mathfrak{p} = T_{\mathsf{c}(o)} N^6 = T_{\mathsf{c}(o)} G/\mathrm{T}^3$, induces an invariant 1-form form $\tilde{\xi}$ and moreover an exact invariant 2-form $\mathrm{d}\,\tilde{\xi}$ on $C_{k,\ell,m}$. Since $C_{\mathfrak{p}}(\mathfrak{h})$ is trivial, there is a bijection

$$\xi \in \mathfrak{m}_0 \longmapsto \tilde{\xi} \in (\Lambda^1 \mathfrak{m}_0^*)^H \cong (\mathfrak{m}_0^*)^H \cong \mathfrak{m}_0^H.$$

This means that $\chi_*(e_9)(e^7) = 0$, $\chi_*(e_9)(e^8) = 0$, i.e., the dual 1-forms e^7, e^8 of e_7, e_8 induce G-invariant 1-forms on $C_{k,\ell,m}$. The same holds true for the dual 1-forms

$$e_\theta^7 = \cos(\theta)\, e^7 + \sin(\theta)\, e^8, \quad e_\theta^8 = -\sin(\theta)\, e^7 + \cos(\theta)\, e^8,$$

and their wedge product satisfies the relation $e_\theta^7 \wedge e_\theta^8 = e^7 \wedge e^8$. On the other hand, (6.2) gives

$$\chi_*(e_9)e^1 = \sum_{j=1}^8 \left(\chi_*(e_9)e^1\right)(e_j)e^j = -\sum_j e^1\left(\chi_*(e_9)e_j\right)e^j = -e^1\left(\chi_*(e_9)e_2\right)e^2 = -ke^2,$$

and, similarly,

$$\chi_*(e_9)e^2 = ke^1, \ \chi_*(e_9)e^3 = -\ell e^4, \ \chi_*(e_9)e^4 = \ell e^3, \ \chi_*(e_9)e^5 = -me^6, \ \chi_*(e_9)e^6 = me^5. \quad (7.2)$$

Thus, a basis for the space of invariant 1-forms on $C_{k,\ell,m}$, when $k > \ell > m > 0$, is given by $\{e^7, e^8\}$.

For the invariant 2-forms on $C_{k,\ell,m}$, we obtain the following.

Lemma 7.4. *If the integers k, ℓ, m satisfy $k > \ell > m > 0$, then the space of G-invariant 2-forms on $C_{k,\ell,m}$ is 4-dimensional and it is generated by the 2-forms $\{e^{12}, e^{34}, e^{56}, e^{78}\}$.*

Proof. By the orthogonal decomposition $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{m}_0 = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_0$, one obtains the orthogonal decomposition

$$(\Lambda^2 \mathfrak{m}^*)^H = \left(\Lambda^2 (\mathfrak{p} \oplus \mathfrak{m}_0)^*\right)^H = (\Lambda^2 \mathfrak{p}^*)^H \oplus (\mathfrak{p}^* \wedge \mathfrak{m}_0^*)^H \oplus (\Lambda^2 \mathfrak{m}_0^*)^H,$$

where $(\Lambda^2 \mathfrak{p}^*)^H = (\Lambda^2 \mathfrak{m}_1^*)^H \oplus (\Lambda^2 \mathfrak{m}_2^*)^H \oplus (\Lambda^2 \mathfrak{m}_3^*)^H \oplus (\mathfrak{m}_1^* \wedge \mathfrak{m}_2^*)^H \oplus (\mathfrak{m}_1^* \wedge \mathfrak{m}_3^*)^H \oplus (\mathfrak{m}_2^* \wedge \mathfrak{m}_3^*)^H$. It is easy to see that the 2-forms e^{12}, e^{34}, e^{56}, and e^{78} are $\mathrm{Ad}(H)$-invariant. In particular

$$(\Lambda^2 \mathfrak{m}_1^*)^H = \mathrm{span}_{\mathbb{R}}\{e^{12}\}, \ (\Lambda^2 \mathfrak{m}_2^*)^H = \mathrm{span}_{\mathbb{R}}\{e^{34}\}, \ (\Lambda^2 \mathfrak{m}_3^*)^H = \mathrm{span}_{\mathbb{R}}\{e^{56}\}, \ (\Lambda^2 \mathfrak{m}_0^*)^H = \mathrm{span}_{\mathbb{R}}\{e^{78}\}.$$

Moreover, an element of the triple (k, ℓ, m) is non-zero if and only if the H-module $(\mathfrak{p}^* \wedge \mathfrak{m}_0^*)^H \cong (\mathfrak{p} \wedge \mathfrak{m}_0)^H$ vanishes. This follows from the relations

$$\begin{array}{llll}
\chi_*(e_9)e^{17} = -ke^{27}, & \chi_*(e_9)e^{47} = \ell e^{37}, & \chi_*(e_9)e^{18} = -ke^{28}, & \chi_*(e_9)e^{48} = \ell e^{38}, \\
\chi_*(e_9)e^{27} = ke^{17}, & \chi_*(e_9)e^{57} = -me^{67}, & \chi_*(e_9)e^{28} = ke^{18}, & \chi_*(e_9)e^{58} = -me^{68}, \\
\chi_*(e_9)e^{37} = -\ell e^{47}, & \chi_*(e_9)e^{67} = me^{57}, & \chi_*(e_9)e^{38} = -\ell e^{48}, & \chi_*(e_9)e^{68} = me^{58}.
\end{array}$$

For the remaining mixed terms corresponding to the modules $(\mathfrak{m}_1^* \wedge \mathfrak{m}_2^*)$, $(\mathfrak{m}_1^* \wedge \mathfrak{m}_3^*)$ and $(\mathfrak{m}_2^* \wedge \mathfrak{m}_3^*)$, one similarly computes

$$\begin{array}{lll}
\chi_*(e_9)e^{13} = -ke^{23} - \ell e^{14}, & \chi_*(e_9)e^{15} = -ke^{25} - me^{16}, & \chi_*(e_9)e^{35} = -\ell e^{45} - me^{36}, \\
\chi_*(e_9)e^{24} = ke^{14} + \ell e^{23}, & \chi_*(e_9)e^{26} = ke^{16} + me^{25}, & \chi_*(e_9)e^{46} = \ell e^{36} + me^{45}, \\
\chi_*(e_9)e^{14} = -ke^{24} + \ell e^{13}, & \chi_*(e_9)e^{16} = -ke^{26} + me^{15}, & \chi_*(e_9)e^{36} = -\ell e^{46} + me^{35}, \\
\chi_*(e_9)e^{23} = ke^{13} - \ell e^{24}, & \chi_*(e_9)e^{25} = ke^{15} - me^{26}, & \chi_*(e_9)e^{45} = \ell e^{35} - me^{46}.
\end{array}$$

Using these results, we see that as long as $k \neq \ell \neq m$ and k, ℓ, m are non-zero, there are no further $\mathrm{Ad}(H)$-invariant 2-forms.   □

Remark 7.5. Up to scaling, the 2-forms e^{12}, e^{34}, e^{56} are the unique $\mathsf{U}(1)$-invariant Kähler forms on the corresponding factors $\mathsf{SU}(2)/\mathsf{U}(1)$ of the product $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. By Proposition 7.4, it follows that the 2-form $\omega_{\mathfrak{m}} := \omega_{\mathfrak{p}} + \omega_{\mathfrak{m}_0} = e^{12} + e^{34} + e^{56} + e^{78}$ is the fundamental 2-form associated to the invariant complex structure $J_{\mathfrak{m}} = J_{\mathfrak{p}} + J_{\mathfrak{m}_0}$ discussed in Remark 7.1, with $\omega_{\mathfrak{m}_0} = e^{78}$. Of course, $\omega_{\mathfrak{m}}$ is not Kähler, since $\mathrm{d}e^{78} \neq 0$ (cf. Appendix B).

We can now discuss the invariant 3-forms on $C_{k,\ell,m}$. As we will see below, the condition $k = \ell + m$ appears in a natural way.

Lemma 7.6. *Assume that the integers k, ℓ, m, with $k > \ell > m > 0$, satisfy $k - \ell - m = 0$. Then, the space of G-invariant 3-forms on $C_{\ell+m,\ell,m}$ is 8-dimensional and it is generated by the 3-forms*

$$e^{127}, \ e^{128}, \ e^{347}, \ e^{348}, \ e^{567}, \ e^{568}, \tag{7.3}$$

and

$$\alpha_1 := e^{135} - e^{146} + e^{245} + e^{236}, \quad \alpha_2 := e^{136} + e^{145} + e^{246} - e^{235}. \tag{7.4}$$

Proof. Since $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{m}_0$, we obtain

$$(\Lambda^3 \mathfrak{m}^*)^H = \left(\Lambda^3 (\mathfrak{p} \oplus \mathfrak{m}_0)^* \right)^H = (\Lambda^3 \mathfrak{p}^*)^H \oplus (\Lambda^2 \mathfrak{p}^* \wedge \mathfrak{m}_0^*)^H \oplus (\mathfrak{p}^* \wedge \Lambda^2 \mathfrak{m}_0^*)^H.$$

The 3-forms given by (7.3) are invariant, independently of the condition $k - \ell - m = 0$. They occur by considering the wedge product of the invariant 2-forms e^{12}, e^{34}, e^{56} with the elements e^7 and e^8, which span $(\mathfrak{m}_0^*)^H \cong (\mathfrak{m}_0)^H = \mathfrak{m}_0$. Thus, they span the 6-dimensional factor $(\Lambda^2 \mathfrak{p}^* \wedge \mathfrak{m}_0^*)^H$. We mention that the invariant 3-forms $e_\theta^{ij7} = e^{ij} \wedge e_\theta^7$ and $e_\theta^{ij8} = e^{ij} \wedge e_\theta^8$, for $ij \in \{12, 34, 56\}$, do not induce new forms. Now, for generic non-zero k, ℓ, m, the third module $(\mathfrak{p}^* \wedge \Lambda^2 \mathfrak{m}_0^*)^H$ vanishes, since

$$\begin{aligned}
\chi_*(e_9)e^{178} = -ke^{278}, & \quad \chi_*(e_9)e^{278} = ke^{178}, & \quad \chi_*(e_9)e^{378} = -\ell e^{478}, \\
\chi_*(e_9)e^{478} = \ell e^{378}, & \quad \chi_*(e_9)e^{578} = -me^{478}, & \quad \chi_*(e_9)e^{678} = me^{578},
\end{aligned}$$

Let us now consider to the first factor. Due to the orthogonal decomposition $\mathfrak{p} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, one has

$$\begin{aligned}
(\Lambda^3 \mathfrak{p}^*)^H = & \ (\Lambda^2 \mathfrak{m}_1^* \wedge \mathfrak{m}_2^*)^H \oplus (\Lambda^2 \mathfrak{m}_1^* \wedge \mathfrak{m}_3^*)^H \oplus (\Lambda^2 \mathfrak{m}_2^* \wedge \mathfrak{m}_3^*)^H \\
& \oplus (\mathfrak{m}_1^* \wedge \Lambda^2 \mathfrak{m}_2^*)^H \oplus (\mathfrak{m}_1^* \wedge \Lambda^2 \mathfrak{m}_3^*)^H \oplus (\mathfrak{m}_2^* \wedge \Lambda^2 \mathfrak{m}_3^*)^H \oplus (\mathfrak{m}_1^* \wedge \mathfrak{m}_2^* \wedge \mathfrak{m}_3^*)^H.
\end{aligned}$$

First, we show that for non-zero k, ℓ, m, the first 6 modules vanish. Indeed, note that

$$\begin{aligned}
\Lambda^2 \mathfrak{m}_1^* \wedge \mathfrak{m}_2^* = \mathrm{span}_{\mathbb{R}} \{e^{123}, e^{124}\}, & \quad \mathfrak{m}_1^* \wedge \Lambda^2 \mathfrak{m}_2^* = \mathrm{span}_{\mathbb{R}} \{e^{134}, e^{234}\}, \\
\Lambda^2 \mathfrak{m}_1^* \wedge \mathfrak{m}_3^* = \mathrm{span}_{\mathbb{R}} \{e^{125}, e^{126}\}, & \quad \mathfrak{m}_1^* \wedge \Lambda^2 \mathfrak{m}_3^* = \mathrm{span}_{\mathbb{R}} \{e^{156}, e^{256}\}, \\
\Lambda^2 \mathfrak{m}_2^* \wedge \mathfrak{m}_3^* = \mathrm{span}_{\mathbb{R}} \{e^{345}, e^{346}\}, & \quad \mathfrak{m}_2^* \wedge \Lambda^2 \mathfrak{m}_3^* = \mathrm{span}_{\mathbb{R}} \{e^{356}, e^{456}\}.
\end{aligned}$$

Computing the action of $\chi_*(e_9)$ on the 3-forms appearing above, we see that for non-zero k, ℓ, m, none of these 3-forms or any linear combination of them belong to the kernel of the isotropy action of H. Let us finally prove that the condition $k - \ell - m = 0$ is equivalent to

$$(\mathfrak{m}_1^* \wedge \mathfrak{m}_2^* \wedge \mathfrak{m}_3^*)^H = \mathrm{span}_{\mathbb{R}} \{\alpha_1, \alpha_2\}.$$

This module is independent of the rotation that one may apply to $\mathfrak{m}_0$. Moreover, $\mathfrak{m}_1^* \wedge \mathfrak{m}_2^* \wedge \mathfrak{m}_3^* = \mathrm{span}_{\mathbb{R}} \{e^{135}, e^{145}, e^{136}, e^{146}, e^{235}, e^{245}, e^{236}, e^{246}\}$ and

$$\chi_*(e_9)(\alpha_1) = (k - \ell - m)\alpha_2, \quad \chi_*(e_9)(\alpha_2) = -(k - \ell - m)\alpha_1,$$

which shows that $\alpha_1, \alpha_2 \in (\Lambda^3 \mathfrak{m}^*)^H$ if and only of $k - \ell - m = 0$. Finally, it is easy to see that these invariant 3-forms constitute a basis of $(\Lambda^3 \mathfrak{m}^*)^H$ if and only if $k = \ell + m$.   □

Combining Lemmas 7.4 and 7.6 allows us to describe the invariant 4-forms on $C_{\ell+m,\ell,m}$.

Lemma 7.7. *Assume that the integers k, ℓ, m, satisfy $k - \ell - m = 0$, with $k > \ell > m > 0$. Then, the space of G-invariant 4-forms on $C_{\ell+m,\ell,m}$ is 10-dimensional, and it is generated by the 4-forms*

$$e^{1234}, \ e^{1256}, \ e^{1278}, \ e^{5678}, \ e^{3478}, \ e^{3456},$$

and

$$\beta_1 := \alpha_1 \wedge e^7, \quad \beta_2 := \alpha_2 \wedge e^8, \quad \zeta_1 := \alpha_2 \wedge e^7, \quad \zeta_2 := -\alpha_1 \wedge e^8$$

where α_1 and α_2 are the invariant 3-forms described in (7.4). Note that β_2 and ζ_2 are obtained by applying the Hodge star operator of the bi-invariant metric $\langle \cdot, \cdot \rangle$ to β_1 and ζ_1, respectively.

Proof. By the splitting $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{m}_0$, we obtain the $\mathrm{Ad}(H)$-invariant orthogonal decomposition

$$(\Lambda^4 \mathfrak{m}^*)^H = \left(\Lambda^4 (\mathfrak{p} \oplus \mathfrak{m}_0)^* \right)^H = (\Lambda^4 \mathfrak{p}^*)^H \oplus (\Lambda^3 \mathfrak{p}^* \wedge \mathfrak{m}_0^*)^H \oplus (\Lambda^2 \mathfrak{p}^* \wedge \Lambda^2 \mathfrak{m}_0^*)^H. \qquad (7.5)$$

The module $(\Lambda^2 \mathfrak{p}^* \wedge \Lambda^2 \mathfrak{m}_0^*)^H$ is generated by the invariant 4-forms e^{1278}, e^{3478} and e^{5678}, which are obtained by wedging the 3-forms given in (7.3) with the generators of $\mathfrak{m}_0$. Consider now the second summand $(\Lambda^3 \mathfrak{p}^* \wedge \mathfrak{m}_0^*)^H$. We can argue in a similar way as we did for $(\Lambda^3 \mathfrak{p}^*)^H$. In detail, we know that $\Lambda^3 \mathfrak{p}^* \wedge \mathfrak{m}_0^*$ splits into a direct sum of seven subspaces, and it is easy to see that any isomorphism $\mathfrak{m}_0^\theta \cong \mathfrak{m}_0$ does not contribute with further summands. An inspection of each subspace allows us to conclude that only one of them contains elements belonging to the kernel of the isotropy action when $k = \ell + m$, namely $\mathfrak{m}_1^* \wedge \mathfrak{m}_2^* \wedge \mathfrak{m}_3^* \wedge \mathfrak{m}_0^*$. More precisely, we have

$$(\mathfrak{m}_1^* \wedge \mathfrak{m}_2^* \wedge \mathfrak{m}_3^* \wedge \mathfrak{m}_0^*)^H = (\mathfrak{m}_1^* \wedge \mathfrak{m}_2^* \wedge \mathfrak{m}_3^*)^H \wedge (\mathfrak{m}_0^*)^H = \mathrm{span}_{\mathbb{R}} \{ \alpha_1 \wedge e^7, \alpha_1 \wedge e^8, \alpha_2 \wedge e^7, \alpha_2 \wedge e^8 \}.$$

This immediately follows from the identities

$$\chi_*(e_9)\beta_1 = (k - \ell - m)\zeta_1, \quad \chi_*(e_9)\beta_2 = (k - \ell - m)\zeta_2,$$
$$\chi_*(e_9)\zeta_1 = -(k - \ell - m)\beta_1, \quad \chi_*(e_9)\zeta_2 = -(k - \ell - m)\beta_2.$$

To conclude the proof, we have to examine the first module $(\Lambda^4 \mathfrak{p}^*)^H$ in (7.5). It decomposes as follows:

$$
\begin{aligned}
(\Lambda^4 \mathfrak{p}^*)^H \ = \ & (\Lambda^2 \mathfrak{m}_1^* \wedge \Lambda^2 \mathfrak{m}_2^*)^H \oplus (\Lambda^2 \mathfrak{m}_1^* \wedge \Lambda^2 \mathfrak{m}_3^*)^H \oplus (\Lambda^2 \mathfrak{m}_2^* \wedge \Lambda^2 \mathfrak{m}_3^*)^H \\
& \oplus (\Lambda^2 \mathfrak{m}_1^* \wedge \mathfrak{m}_2^* \wedge \mathfrak{m}_3^*)^H \oplus (\mathfrak{m}_1^* \wedge \Lambda^2 \mathfrak{m}_2^* \wedge \mathfrak{m}_3^*)^H \oplus (\mathfrak{m}_1^* \wedge \mathfrak{m}_2^* \wedge \Lambda^2 \mathfrak{m}_3^*)^H .
\end{aligned}
$$

where the first three modules are 1-dimensional and generated by $e^{1234}, e^{1256}, e^{3456}$, respectively. A direct computation shows that the remaining modules are trivial. $\qquad \square$

From the above proposition, we obtain the following.

Corollary 7.8. *Consider the homogeneous spaces $C_{k,\ell,m}$ with $k > \ell > m > 0$. Then, the 4-form*

$$
\begin{aligned}
\Phi \ = \ & e^{1234} + e^{1256} - e^{1278} + e^{1357} + e^{1368} + e^{1458} - e^{1467} \\
& + e^{5678} + e^{3478} - e^{3456} + e^{2468} + e^{2457} - e^{2358} + e^{2367} \qquad (7.6)
\end{aligned}
$$

is invariant if and only if $k - \ell - m = 0$. Whenever this condition is satisfied, Φ induces an invariant Spin(7)-structure on $C_{\ell+m,\ell,m}$.

Remark 7.9. Due to the results of Appendix B, it is straightforward to check that there are no invariant closed 1-forms on $C_{k,\ell,m}$ as long as the integers k, ℓ, m satisfy $k > \ell > m > 0$. Thus, this homogeneous space cannot admit any invariant l.c.p. Spin(7)-structure.

We now describe a 5-parameter family of invariant Spin(7)-structures on $C_{\ell+m,\ell,m}$ inducing the most general invariant metric $(\ ,\ )_{y_1,y_2,y_3,y_4,y_5} := (\ ,\ )_{y_1,y_2,y_3,y_4,y_5,0}$ adapted to the reductive decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_0$, with $\mathfrak{m}_0 = \mathfrak{m}_4 \oplus \mathfrak{m}_5$ (cf. Proposition 7.3). The choice $\theta = 0$ for the summand $\mathfrak{m}_0^\theta$ is just a matter of convenience, as it simplifies the computations afterwards.

Proposition 7.10. *The 4-form*

$$\Phi_{y_1,y_2,y_3,y_4,y_5} := y_1^2 y_2^2 e^{1234} + y_1^2 y_3^2 e^{1256} - y_2^2 y_3^2 e^{3456} - y_1^2 y_4 y_5 e^{1278} + y_2^2 y_4 y_5 e^{3478} + y_3^2 y_4 y_5 e^{5678}$$
$$+ y_1 y_2 y_3 y_4 \, \beta_1 + y_1 y_2 y_3 y_5 \, \beta_2,$$

where $y_1, \ldots, y_5$, are positive real parameters, defines an invariant $\mathsf{Spin}(7)$*-structure on $C_{\ell+m,\ell,m}, \ell > m > 0$. It induces the metric $(\ ,\ )_{y_1,y_2,y_3,y_4,y_5}$, and its Lee form is given by*

$$\vartheta_{y_1,y_2,y_3,y_4,y_5} = \frac{2}{7\bar{c}_7}\Big(\ell m(y_2^2 y_3^2 - y_1^2 y_3^2 + 6y_1^2 y_2^2)y_4^2 y_5^2 + m^2(y_3^2 y_4^2 + y_1^2 y_4^2 + 4y_1^2 y_3^2)y_2^2 y_5^2$$
$$+ 2\ell^2(y_4^2 + 2y_3^2)y_1^2 y_2^2 y_5^2\Big)e^8 - \frac{2(2l+m)}{7\bar{c}_8}\Big((2y_1^2 + y_5^2)y_2^2 y_3^2 y_4^2\Big)e^7,$$

where $\bar{c}_7$ and $\bar{c}_8$ are obtained by setting $k = \ell + m$ in the expressions of c_7 and c_8, respectively. In particular, the $\mathsf{Spin}(7)$*-structure is always of mixed type.*

Proof. It is clear that the 4-form $\Phi_{y_1,y_2,y_3,y_4,y_5}$ is invariant and admissible and that it induces the metric $(\ ,\ )_{y_1,\ldots,y_5}$. For brevity, we denote by $\star$ the corresponding Hodge operator. Recall that the Lee form is given by

$$\vartheta_{y_1,y_2,y_3,y_4,y_5} = -\frac{1}{7}\star(\star d\Phi_{y_1,y_2,y_3,y_4,y_5} \wedge \Phi_{y_1,y_2,y_3,y_4,y_5}).$$

Using the results of Appendix B together with the definition of the Hodge operator, we obtain the expression of $\vartheta_{y_1,y_2,y_3,y_4,y_5}$ described above. To determine the Fernández type of this $\mathsf{Spin}(7)$-structure, it is sufficient to observe that the Lee form is never closed (cf. Remark 7.9), and that the condition $\vartheta_{y_1,y_2,y_3,y_4,y_5} = 0$ is equivalent to a system of two polynomial equations in the variables $y_1, \ldots, y_5, \ell, m$, which has no solutions under the constraints $\ell > m > 0$ and $y_i > 0$, $i = 1, \ldots, 5$. Thus, the $\mathsf{Spin}(7)$-structure is of mixed type. $\qquad\square$

Corollary 7.11. *The invariant* $\mathsf{Spin}(7)$*-structure defined by the admissible 4-form $\Phi := \Phi_{1,\ldots,1}$ on $C_{\ell+m,\ell,m}$, with $\ell > m > 0$, is of mixed type and its characteristic connection ∇ coincides with the canonical connection ∇^0 with respect to the naturally reductive structure induced by $g := (\ ,\ )_{1,\ldots,1}$. In particular, its torsion form is parallel, i.e., $\nabla T = 0$.*

Proof. We already know that the 4-form Φ defines a $\mathsf{Spin}(7)$-structure of mixed type and it induces the normal metric $g := (\ ,\ )_{1,\ldots,1}$. The homogeneous space $(C_{k,\ell,m}, g)$ is naturally reductive. Consequently, the canonical connection ∇^0 has totally skew-symmetric torsion (cf. [33]). Moreover, since Φ and g are G-invariant, they are both parallel with respect to ∇^0 (see e.g. [20, Ch. X, Prop. 2.7]). Thus, the connection ∇^0 must coincide with the canonical connection ∇ by the uniqueness of the latter [15, Thm. 1.1], and the G-invariant torsion form T is ∇-parallel.

Using the results of Appendix B, we can easily compute the expression of the torsion form, obtaining

$$T = \star d\,\Phi - \frac{7}{6}\star(\vartheta \wedge \Phi) = -\frac{m(\ell+m)}{\bar{c}_7}e^{127} - \frac{\ell}{\bar{c}_8}e^{128} - \frac{\ell m}{\bar{c}_7}e^{347} + \frac{\ell+m}{\bar{c}_8}e^{348} + \frac{(\ell+m)^2+\ell^2}{\bar{c}_7}e^{567},$$

where $\bar{c}_7$ and $\bar{c}_8$ are obtained by setting $k = \ell + m$ in the expressions of c_7 and c_8, respectively. $\quad\square$

7.2. The Calabi-Eckmann manifold $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$.

Let $G := \mathsf{SU}(3) \times \mathsf{SU}(2)$ and $H := \mathsf{SU}(2)$. $M = G/H$ is a direct product of prime homogeneous spaces, hence a reductive decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$ is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{n} \oplus \mathbb{R}^3.$$

Clearly, $\mathfrak{n}$ coincides with the tangent space of $\mathsf{SU}(3)/\mathsf{SU}(2)$ at the identity coset, and $\mathbb{R}^3 \cong T_e\mathsf{SU}(2)$. Let us now consider the following basis of $\mathfrak{su}(3)$

$$e_1 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_4 := \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$e_5 := \frac{1}{\sqrt{3}} \begin{pmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad u_6 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad u_7 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad u_8 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}.$$

With this choice, the isotropy subalgebra $\mathfrak{h}$ is generated by the triple $\{u_6, u_7, u_8\}$, and $\{e_1, \ldots, e_5\}$ is a basis of $\mathfrak{n} = \mathsf{V} \oplus \mathbb{R}$, where the module $\mathbb{R}$ is spanned by e_5 and $\mathsf{V} \cong \mathbb{H} = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4\} \cong \mathrm{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Moreover, it is easy to check that this $\mathfrak{su}(3)$-basis is orthonormal with respect to the bi-invariant inner product

$$\langle X, Y \rangle := -\frac{1}{2}\operatorname{tr}(XY). \tag{7.7}$$

From now on, we shall identify $\mathsf{V} \cong \mathsf{V}^*$ via the quaternionic metric on $\mathsf{V} \cong \mathbb{H}$. A basis for the tangent space $\mathfrak{m}$ is given by the union of $\{e_1, \ldots, e_5\}$ with a basis $\{e_6, e_7, e_8\}$ of $T_e\mathsf{SU}(2) \cong \mathfrak{su}(2)$. The latter may be chosen as follows

$$e_6 := -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_7 := \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_8 := \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

so that $[e_6, e_7] = e_8$, $[e_6, e_8] = -e_7$, $[e_7, e_8] = e_6$. Since $\mathfrak{m} = \mathfrak{n} \oplus \mathbb{R}^3$ and the action of H on $\mathsf{SU}(2) \cong \mathbb{S}^3$ is trivial, we have

$$[e_i, e_j] = 0 = [u_k, e_j], \quad \forall\, i = 1, \ldots 5, \ k, j = 6, 7, 8.$$

Of course, e_5 is also invariant under the isotropy action, i.e., $[u_k, e_5] = 0$ for any $k = 6, 7, 8$, while on V we have

$$\begin{array}{llll} \chi_*(u_6)e_1 = e_2, & \chi_*(u_6)e_2 = -e_1, & \chi_*(u_6)e_3 = e_4, & \chi_*(u_6)e_4 = -e_3, \\ \chi_*(u_7)e_1 = e_3, & \chi_*(u_7)e_2 = -e_4, & \chi_*(u_7)e_3 = -e_1 & \chi_*(u_7)e_4 = e_2, \\ \chi_*(u_8)e_1 = e_4, & \chi_*(u_8)e_2 = e_3, & \chi_*(u_8)e_3 = -e_2, & \chi_*(u_8)e_4 = -e_1. \end{array}$$

Hence, in terms of skew-symmetric matrices E_{ij} we obtain the orthogonal transformations

$$\chi_*(u_6)|_{\mathfrak{m}} = E_{12} + E_{34}, \quad \chi_*(u_7)|_{\mathfrak{m}} = E_{13} - E_{24}, \quad \chi_*(u_8)|_{\mathfrak{m}} = E_{14} + E_{23}.$$

We are now ready to describe the space of G-invariant forms on M. As before, we shall denote by $e^k \in \mathfrak{m}^*$ the dual of e_k.

Lemma 7.12. *The spaces of G-invariant k-forms on $M = (\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$, for $k = 1, 2, 3, 4$, have the following dimensions:*

$$\dim\left((\Lambda^1\mathfrak{m}^*)^H\right) = 4, \quad \dim\left((\Lambda^2\mathfrak{m}^*)^H\right) = 9, \quad \dim\left((\Lambda^3\mathfrak{m}^*)^H\right) = 16, \quad \dim\left((\Lambda^4\mathfrak{m}^*)^H\right) = 20.$$

A basis of each space is described in the proof.

Proof. From the above description of the isotropy action, we immediately see that $(\Lambda^1\mathfrak{m}^*)^H$ is spanned by $\{e^5, e^6, e^7, e^8\}$.

Let us consider the space of invariant 2-forms. Since $\mathfrak{m} = \mathfrak{n} \oplus \mathbb{R}^3$ is an orthogonal $\mathrm{Ad}(H)$-invariant decomposition, we obtain the invariant splitting

$$(\Lambda^2\mathfrak{m}^*)^H = (\Lambda^2\mathfrak{n}^*)^H \oplus (\Lambda^1\mathfrak{n}^* \wedge \mathbb{R}^3) \oplus (\Lambda^2\mathbb{R}^3)^H,$$

where we identify $\mathbb{R}^3 \cong (\mathbb{R}^3)^*$. Obviously, the second module is spanned by $\{e^{56}, e^{57}, e^{58}\}$, and the third one by $\{e^{67}, e^{68}, e^{78}\}$. Thus we are left with

$$(\Lambda^2 \mathfrak{n}^*)^H = (\Lambda^2 \mathsf{V})^H \oplus (\mathsf{V} \wedge \mathbb{R})^H \, .$$

The summand $(\mathsf{V} \wedge \mathbb{R})^H$ is easily seen to be trivial, while for the first summand we recall that

$$\Lambda^2 \mathsf{V} = \Lambda^2 \mathbb{H} \cong \mathfrak{so}(4) = \mathfrak{sp}(1)^+ \oplus \mathfrak{sp}(1)^- = \mathrm{span}_{\mathbb{R}}\{e^{12}, e^{13}, e^{14}\} \oplus \mathrm{span}_{\mathbb{R}}\{e^{23}, e^{24}, e^{34}\} \, .$$

Then, a direct computation shows that a basis of $(\Lambda^2 \mathsf{V})^H$ is given by $\{e^{12} - e^{34},\ e^{13} + e^{24},\ e^{14} - e^{23}\}$.

Let us now consider the space of invariant 3-forms

$$(\Lambda^3 \mathfrak{m}^*)^H = (\Lambda^3 \mathfrak{n}^*)^H \oplus (\Lambda^2 \mathfrak{n}^* \wedge \mathbb{R}^3)^H \oplus (\Lambda^1 \mathfrak{n}^* \wedge \Lambda^2 \mathbb{R}^3)^H \oplus (\Lambda^3 \mathbb{R}^3)^H \, .$$

The last two summands are generated by $\{e^{567}, e^{568}, e^{578}\}$ and $\{e^{678}\}$, respectively. The second summand is spanned by $\{e^{12k} - e^{34k},\ e^{13k} + e^{24k},\ e^{14k} - e^{23k}\ :\ k = 6, 7, 8\}$. Thus, it has dimension 9. Finally, we have

$$(\Lambda^3 \mathfrak{n}^*)^H = (\Lambda^3 \mathsf{V})^H \oplus (\Lambda^2 \mathsf{V} \wedge \mathbb{R})^H \, ,$$

where the first summand is trivial, while a basis of invariant 3-forms for the second summand is given by $\{e^{125} - e^{345},\ e^{135} + e^{245},\ e^{145} - e^{235}\}$. Summing up, the space of invariant 3-forms on M^8 is 16-dimensional.

As for the H-module $(\Lambda^4 \mathfrak{m}^*)^H$, we have

$$(\Lambda^4 \mathfrak{m}^*)^H = (\Lambda^4 \mathfrak{n}^*)^H \oplus (\Lambda^3 \mathfrak{n}^* \wedge \mathbb{R}^3)^H \oplus (\Lambda^2 \mathfrak{n}^* \wedge \Lambda^2 \mathbb{R}^3)^H \oplus (\Lambda^1 \mathfrak{n}^* \wedge \Lambda^3 \mathbb{R}^3)^H \, .$$

The second and the third module are both 9-dimensional. A basis of $(\Lambda^3 \mathfrak{n}^* \wedge \mathbb{R}^3)^H \cong (\Lambda^3 \mathfrak{n}^*)^H \wedge (\mathbb{R}^3)^H$ is given by

$$\left\{ e^{125k} - e^{345k},\ e^{135k} + e^{245k},\ e^{145k} - e^{235k}\ :\ k = 6, 7, 8 \right\},$$

and a basis of $(\Lambda^2 \mathfrak{n}^* \wedge \Lambda^2 \mathbb{R}^3)^H \cong (\Lambda^2 \mathfrak{n}^*)^H \wedge (\Lambda^2 \mathbb{R}^3)^H$ is

$$\left\{ (e^{12} - e^{34}) \wedge \alpha,\ (e^{13} + e^{24}) \wedge \alpha,\ (e^{14} - e^{23}) \wedge \alpha\ :\ \alpha = e^{67}, e^{68}, e^{78} \right\}$$

The fourth module is 1-dimensional with generator $\{e^{5678}\}$. Also, $(\Lambda^4 \mathfrak{n}^*)^H \cong (\Lambda^4 \mathsf{V})^H \oplus (\Lambda^3 \mathsf{V} \wedge \mathbb{R})^H$ where $\{e^{1234}\}$ spans the first summand, while the second one is trivial. This finishes the proof. $\square$

Using the basis of invariant 4-forms, we obtain the following.

Corollary 7.13. *The 4-form*

$$\begin{aligned}
\Phi\ :=\ & e^{1234} + e^{1256} - e^{1278} + e^{1357} + e^{1368} + e^{1458} - e^{1467} \\
& + e^{5678} + e^{3478} - e^{3456} + e^{2468} + e^{2457} - e^{2358} + e^{2367}
\end{aligned}$$

induces a G-invariant $\mathsf{Spin}(7)$-structure on $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$.

Remark 7.14. Using the Koszul formula, it is straightforward to check that there are no G-invariant closed 1-forms on $M = G/H$. Thus, we deduce that the Calabi-Eckmann manifold $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$ cannot admit any invariant l.c.p. $\mathsf{Spin}(7)$-structure.

Now, we shall construct a family of invariant $\mathsf{Spin}(7)$-structures inducing the general invariant metric on $M = G/H$. To this aim, we first study the space of G-invariant Riemannian metrics on M or, equivalently, the space of $\mathrm{Ad}(H)$-invariant inner products on the reductive complement $\mathfrak{m}$. Recall that the decomposition of $\mathfrak{m}$ into inequivalent irreducible H-modules is given by $\mathfrak{m} = \mathsf{V} \oplus \mathbb{R} \oplus \mathbb{R}^3$, with H acting trivially on $\mathbb{R}^3 \cong \mathfrak{su}(2)$. Consequently, an $\mathrm{Ad}(H)$-invariant inner product g on $\mathfrak{m}$ must be of the form

$$g = t_1^2 \langle \cdot, \cdot \rangle|_{\mathsf{V}} + t_2^2 \langle \cdot, \cdot \rangle|_{\mathbb{R}} + h,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathfrak{su}(3)$ given in (7.7), $t_1, t_2 \in \mathbb{R}^+$, and h is an inner product on $\mathbb{R}^3 \cong \mathfrak{su}(2)$ corresponding to a left-invariant Riemannian metric on $\mathsf{SU}(2)$. It follows from [24]

that there exist suitable real parameters $0 < t_3 \leq t_4 \leq t_5$ for which $(\mathsf{SU}(2), h)$ is isometrically isomorphic to $\mathsf{SU}(2)$ endowed with the left-invariant Riemannian metric induced by the following inner product on $\mathfrak{su}(2)$

$$h_{t_3,t_4,t_5} := t_3^2 \left(e^6 \otimes e^6 \right) + t_4^2 \left(e^7 \otimes e^7 \right) + t_5^2 \left(e^8 \otimes e^8 \right).$$

In particular, the canonical bi-invariant metric on $\mathsf{SU}(2)$ corresponds to $h_{1,1,1}$. The general invariant metric on M is then given by the inner product

$$g_{t_1,t_2,t_3,t_4,t_5} := t_1^2 \left\langle \cdot, \cdot \right\rangle|_{V^4} + t_2^2 \left\langle \cdot, \cdot \right\rangle|_{\mathbb{R}} + h_{t_3,t_4,t_5}.$$

In a similar way as we did for $C_{k,\ell,m}$, it is now possible to show the following.

Proposition 7.15. *On the Calabi-Eckmann manifold* $(\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$, *the invariant* Spin(7)-*structure given by*

$$
\begin{aligned}
\Phi_{t_1,t_2,t_3,t_4,t_5} \quad := \quad & t_1^4 \, e^{1234} + t_1^2 t_2 t_3 \left(e^{1256} - e^{3456} \right) + t_1^2 t_4 t_5 (e^{3478} - e^{1278}) + t_1^2 t_2 t_4 (e^{1357} + e^{2457}) \\
& + t_1^2 t_3 t_5 (e^{1368} + e^{2468}) + t_1^2 t_2 t_5 (e^{1458} - e^{2358}) + t_1^2 t_3 t_4 (e^{2367} - e^{1467}) \\
& + t_2 t_3 t_4 t_5 e^{5678},
\end{aligned}
$$

induces the general invariant metric $g_{t_1,\ldots,t_5}$, *and it is of mixed type with Lee form*

$$\vartheta_{t_1,t_2,t_3,t_4,t_5} = -\frac{1}{7} \left(\frac{2 \, t_2 \left(t_3^2 + t_4^2 + t_5^2 \right)}{t_3 t_4 t_5} \, e^5 + \frac{4\sqrt{3} \, t_3 \left(2 t_1^2 + t_2^2 \right)}{t_1^2 t_2} \, e^6 \right).$$

Arguing as in the proof of Corollary 7.11, the next result immediately follows.

Corollary 7.16. *Let Φ denote the invariant* Spin(7)-*structure on $G/H = (\mathsf{SU}(3)/\mathsf{SU}(2)) \times \mathsf{SU}(2)$ obtained by setting $t_i = 1$, for all $i = 1, \ldots, 5$. Then, the homogeneous space G/H endowed with the metric induced by Φ is naturally reductive, and its canonical connection ∇^0 coincides with the characteristic connection ∇ of Φ. In particular, the torsion of ∇ is given by*

$$T = -\sqrt{3} \left(e^{125} - e^{345} \right) + e^{678},$$

and it is ∇-parallel.

APPENDIX A. PRESENTATIONS FOR $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$ NOT INCLUDED IN THE FAMILY $C_{k,\ell,m}$.

The 8-manifold $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$ has several different presentations as homogeneous space. For example, in Section 6.1 we analysed the pair $(\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{h} = \mathfrak{u}(1)_{k,\ell,m})$ which induces the family $C_{k,\ell,m}$. As a manifold, $C_{k,\ell,m}$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$, and any homogeneous space of the form $(\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2))/\mathsf{U}(1)$ is equivariantly diffeomorphic to $C_{k,\ell,m}$. We also discussed some special cases in Examples 6.2 and 6.3.

In this appendix, our goal is to discuss the presentations of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$ which are not included in the family $C_{k,\ell,m}$ and which are still (almost) effective and simply connected.

Since the symmetric spaces $(\mathsf{SO}(4)/\mathsf{SO}(3)) \times (\mathsf{SO}(4)/\mathsf{SO}(3)) \times (\mathsf{SO}(3)/\mathsf{SO}(2))$ and $(\mathsf{SU}(2) \times \mathsf{SU}(2)/\Delta \mathsf{SU}(2)) \times (\mathsf{SU}(2) \times \mathsf{SU}(2)/\Delta \mathsf{SU}(2)) \times (\mathsf{SU}(2)/\mathsf{U}(1))$ coincide, the isometry group of the symmetric Riemannian product $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$ is 15-dimensional. Consequently, we can focus on compact Lie algebras $\mathfrak{g}$ with smaller dimension. Moreover, as we are interested in invariant Spin(7)-structures, we restrict our attention to the case $\mathrm{rk}\,\mathfrak{h} \leq 3 = \mathrm{rk}\,\mathfrak{spin}(7)$. In Table 2, we list all non-symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ satisfying the above constraints and which are different from $(\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{u}(1)_{k,\ell,m})$.

Remark A.1. In addition to the pairs considered in Table 2, one may also consider a fifth case with $\dim \mathfrak{g} = 14$, i.e., $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. However, this corresponds to the isotropy algebra $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, which has rank bigger than 3.

	dim $\mathfrak{g}$	$\mathfrak{g}$	$\mathfrak{h}$	rk $\mathfrak{h}$	dim $\mathfrak{h}$
(i)	10	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{u}(1) \oplus \mathfrak{u}(1)$	2	2
(ii)	11	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$	$\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$	3	3
(iii)	12	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\Delta\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	2	4
(iv)	13	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$	3	5

TABLE 2. Pairs $(\mathfrak{g}, \mathfrak{h})$ inducing non-symmetric homogeneous spaces covered by $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$.

To obtain the list appearing in Table 2, one has to consider pairs $(\mathfrak{g}, \mathfrak{h})$ of the form

$$\mathfrak{g} = \mathfrak{su}(2)^{\oplus a} \oplus \mathfrak{u}(1)^{\oplus p} \oplus \mathfrak{su}(2) \cong \mathfrak{su}(2)^{\oplus (a+1)} \oplus \mathfrak{u}(1)^{\oplus p},$$
$$\mathfrak{h} = \mathfrak{su}(2)^{\oplus b} \oplus \mathfrak{u}(1)^{\oplus q} \oplus \mathfrak{u}(1) \cong \mathfrak{su}(2)^{\oplus b} \oplus \mathfrak{u}(1)^{\oplus (q+1)},$$

with $a = b+2$, $p = q$, and such that the extra factor $\mathfrak{u}(1)$ of $\mathfrak{h}$ sits diagonally inside the extra factor $\mathfrak{su}(2)$ in $\mathfrak{g}$ (this always induces $\mathbb{S}^2$). Although $p = q$, here we use different indices for the summands $\mathfrak{u}(1)^{\oplus p}$ and $\mathfrak{u}(1)^{\oplus q}$ to emphasize that the abelian factor of $\mathfrak{h}$ does not coincide with the abelian factor of $\mathfrak{g}$. The pairs appearing in Table 2 correspond to the following values of the parameters

Case (i) $a = 2,\ b = 0,\ p = q = 1,$ Case (ii) $a = 2,\ b = 0,\ p = q = 2,$
Case (iii) $a = 3,\ b = 1,\ p = q = 0,$ Case (iv) $a = 3,\ b = 1,\ p = q = 1.$

Let us examine Case (i) in detail. Here, the first factor of $\mathfrak{h}$ sits diagonally inside $\mathfrak{t}^2 \oplus \mathfrak{u}(1)$, where $\mathfrak{t}^2$ is a maximal torus of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In this case, the pair $(\mathfrak{g}, \mathfrak{h})$ is almost effective and it induces the coset $G/H = \frac{\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{U}(1)}{\mathsf{U}(1)} \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)}$, which is simply connected. To see this, it is sufficient to prove that the space $M_1 := (\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{U}(1))/\mathsf{U}(1)$ is simply connected, as G/H is the product of this coset with the simply connected space $M_2 := \mathsf{SU}(2)/\mathsf{U}(1)$. The principal circle bundle

$$\mathsf{U}(1) \xrightarrow{j} \mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{U}(1) \xrightarrow{\pi} M_1 = (\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{U}(1))/\mathsf{U}(1)$$

induces the exact sequence

$$\pi_1(\mathsf{U}(1)) \xrightarrow{j_\sharp} \pi_1(\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{U}(1)) \xrightarrow{\pi_\sharp} \pi_1(M_1) \xrightarrow{\partial} \pi_0(\mathsf{U}(1)) = \{1\},$$

which reduces to $\mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \xrightarrow{\pi_\sharp} \pi_1(M_1) \xrightarrow{\partial} 1$. Since $\ker \pi_\sharp = \mathrm{Im} j_\sharp = \mathbb{Z}$, the map $\pi_\sharp$ must be trivial, whence $\pi_1(M_1) = \ker \partial = \mathrm{Im} \pi_\sharp = \{1\}$. Thus, M_1 is simply connected.

Let $G_1 := \mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{U}(1)$ and $H_1 := \mathsf{U}(1)$, so that $M_1 = G_1/H_1$. Consider the universal covering $\tilde{\pi} : \tilde{G}_1 \to G_1$ of G_1 and $\tilde{H}_1 = \tilde{\pi}^{-1}(H_1)$. Then, we obtain the equivariant diffeomorphism $\tilde{G}_1/\tilde{H}_1 \cong G_1/H_1$, from which it follows that $\tilde{G}_1/\tilde{H}_1 = \mathsf{SU}(2) \times \mathsf{SU}(2)$, i.e., G_1/H_1 is covered by $\mathbb{S}^3 \times \mathbb{S}^3 = \mathsf{SU}(2) \times \mathsf{SU}(2)$. Consequently, G/H is covered by $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$. More generally, we have the following.

Proposition A.2. *Any simply connected coset G/H induced by a pair $(\mathfrak{g}, \mathfrak{h})$ in Table 2, is covered by $\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathbb{S}^2$, which is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$.*

APPENDIX B. DETAILS ON THE INFINITE FAMILY $C_{k,\ell m}$

Here, we collect some useful computational details related to the family $C_{k,\ell,m}$, with k, ℓ, m coprime integers satisfying $k \geq \ell \geq m \geq 0$ and $k > 0$. The notations used in this appendix are those introduced in Sections 6.1 and 7.1.

Let $\tilde{e}_9 := \frac{1}{c_9} e_9$, where $c_9 := \sqrt{(k^2 + \ell^2 + m^2)}$, and let $\{e_1, \ldots, e_8\}$ be the $\langle \cdot, \cdot \rangle$-orthonormal basis of $\mathfrak{m}$ given in (6.1). The non-zero Lie brackets of the basis vectors are the following

$$[e_1, e_2] = -\frac{km}{c_7} e_7 - \frac{\ell}{c_8} e_8 - \frac{k}{c_9} \tilde{e}_9, \quad [e_1, e_7] = \frac{km}{c_7} e_2, \quad [e_1, e_8] = \frac{\ell}{c_8} e_2,$$

$$[e_3, e_4] = -\frac{\ell m}{c_7} e_7 + \frac{k}{c_8} e_8 - \frac{\ell}{c_9} \tilde{e}_9, \quad [e_2, e_7] = -\frac{km}{c_7} e_1, \quad [e_2, e_8] = -\frac{\ell}{c_8} e_1,$$

$$[e_5, e_6] = \frac{(k^2 + \ell^2)}{c_7} e_7 - \frac{m}{c_9} \tilde{e}_9, \quad [e_3, e_7] = \frac{\ell m}{c_7} e_4, \quad [e_3, e_8] = -\frac{k}{c_8} e_4,$$

$$[e_5, e_7] = -\frac{(k^2 + \ell^2)}{c_7} e_6, \quad\quad\quad [e_4, e_7] = -\frac{\ell m}{c_7} e_3, \quad [e_4, e_8] = \frac{k}{c_8} e_3,$$

$$[e_6, e_7] = \frac{(k^2 + \ell^2)}{c_7} e_5,$$

where $c_7 := \sqrt{(k^2 + \ell^2)(k^2 + \ell^2 + m^2)}$, and $c_8 := \sqrt{(k^2 + \ell^2)}$. Moreover, the brackets $[e_9, e_i]$, for $i = 1, \ldots, 8$, are given in (6.2).

The differentials of the dual covectors $\{e^1, \ldots, e^8\}$ of the basis vectors $\{e_1, \ldots, e_8\}$ can be computed using the Koszul formula. In detail, for all $X, Y \in \mathfrak{m}$ we have

$$\mathrm{d}e^i(X, Y) = -e^i([X, Y]_{\mathfrak{m}}),$$

whence we obtain

$$\mathrm{d}e^1 = \frac{km}{c_7} e^{27} + \frac{\ell}{c_8} e^{28}, \quad \mathrm{d}e^2 = -\frac{km}{c_7} e^{17} - \frac{\ell}{c_8} e^{18}, \quad \mathrm{d}e^3 = \frac{\ell m}{c_7} e^{47} - \frac{k}{c_8} e^{48},$$

$$\mathrm{d}e^4 = -\frac{\ell m}{c_7} e^{37} + \frac{k}{c_8} e^{38}, \quad \mathrm{d}e^5 = -\frac{(k^2 + \ell^2)}{c_7} e^{67}, \quad \mathrm{d}e^6 = \frac{(k^2 + \ell^2)}{c_7} e^{57},$$

$$\mathrm{d}e^7 = \frac{km}{c_7} e^{12} + \frac{\ell m}{c_7} e^{34} - \frac{(k^2 + \ell^2)}{c_7} e^{56}, \quad \mathrm{d}e^8 = \frac{\ell}{c_8} e^{12} - \frac{k}{c_8} e^{34}.$$

Using these expressions together with the properties of the differential operator d, it is possible to compute the exterior derivative of every invariant differential form on $C_{k, \ell, m}$.

References

[1] D. V. Alekseevksy, I. Chrysikos. Spin structures on compact homogeneous pseudo-Riemannian manifolds. *Transformation Groups*, in press, https://doi.org/10.1007/s00031-018-9498-1 (cited on p. 13, 15)

[2] D. V. Alekseevksy, I. Chrysikos, A. Taghavi-Chabert. Decomposable $(4, 7)$ solutions in 11-dimensional supergravity. *Classical Quantum Gravity*, **36**(7), 2019. (cited on p. 2)

[3] D. V. Alekseevsky, B. N. Kimel'fel'd. Structure of homogeneous Riemannian spaces with zero Ricci curvature. *Funkcional. Anal. i Priložen.*, **9**(2), 5–11, 1975. (cited on p. 5)

[4] R. M. Arroyo and R. A. Lafuente. The Alekseevskii conjecture in low dimensions. *Math. Ann.*, **367**(1-2), 283–309, 2017. (cited on p. 6)

[5] H. Baum, I. Kath. Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds. *Ann. Global Anal. Geom.*, **17**(1), 1–17, 1999. (cited on p. 9)

[6] A. L. Besse. Einstein manifolds. Volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1987. (cited on p. 5)

[7] C. Böhm. Homogeneous Einstein metrics and simplicial complexes. *J. Differential Geom.*, **67**(1), 79–165, 2004. (cited on p. 5)

[8] C. Böhm, M. M. Kerr. Low-dimensional homogeneous Einstein manifolds. *Trans. Amer. Math. Soc.*, **358**(4), 1455–1468, 2006. (cited on p. 1, 2, 5, 6)

[9] R. L. Bryant. Metrics with exceptional holonomy. *Ann. of Math.*, **126**(3), 525–576, 1987. (cited on p. 3)

[10] F. M. Cabrera. On Riemannian manifolds with Spin(7)-structure. *Publ. Math. Debrecen*, **46**(3-4), 271–283, 1995. (cited on p. 4)

[11] M. Cahen, S. Gutt. Spin structures on compact simply connected Riemannian symmetric spaces. In *Proceedings of the Workshop on Clifford Algebra, Clifford Analysis and their Applications in Mathematical Physics (Ghent, 1988)*, vol. 62, pp. 209–242, 1988. (cited on p. 2, 8)

[12] M. Fernández. A classification of Riemannian manifolds with structure group Spin(7). *Ann. Mat. Pura Appl.*, **143**, 101–122, 1986. (cited on p. 1, 2, 4, 7, 8)

[13] G. R. Jensen. The scalar curvature of left-invariant Riemannian metrics. *Indiana Univ. Math. J.*, **20**, 1125–1144, 1970/1971. (cited on p. 2)

[14] N. Hitchin. Stable forms and special metrics. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, vol. 288 of *Contemp. Math.*, pp. 70–89. Amer. Math. Soc., Providence, RI, 2001. (cited on p. 3)

[15] S. Ivanov. Connections with torsion, parallel spinors and geometry of Spin(7) manifolds. *Math. Res. Lett.*, **11**(2-3), 171–186, 2004. (cited on p. 1, 3, 4, 18)

[16] D. D. Joyce. Compact 8-manifolds with holonomy Spin(7). *Invent. Math.*, **123**(3), 507–552, 1996. (cited on p. 1, 3)

[17] S. Karigiannis. Deformations of G_2 and Spin(7) structures. *Canad. J. Math.*, **57**(5), 1012–1055, 2005. (cited on p. 3, 4, 13)

[18] S. Klaus. Einfach-zusammenhängende Kompakte Homogene Räume bis zur Dimension Neun. Diplomarbeit am Fachbereich Mathematik, Johannes Gutenberg Universität Mainz, 1988. (cited on p. 2, 6, 11, 15)

[19] M. Kervaire. Courbure intégrale généralisée et homotopie. *Math. Ann.*, **131**, 219–252, 1956. (cited on p. 7)

[20] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. II.* Interscience Publishers, New York-London, 1969. (cited on p. 18)

[21] H. B. Lawson, Jr., M.-L. Michelsohn. *Spin geometry*, vol. 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989. (cited on p. 1, 3)

[22] H. V. Lê, M. Munir. Classification of compact homogeneous spaces with invariant G_2-structures. *Adv. Geom.*, **12**(2), 302–328, 2012. (cited on p. 2, 5)

[23] L. Martín-Merchán. Spinorial classification of Spin(7) structures. arXiv:1803.08734 (cited on p. 1)

[24] J. Milnor. Curvatures of left invariant metrics on Lie groups. *Advances in Math.*, **21**(3), 293–329, 1976. (cited on p. 20)

[25] D. Montgomery. Simply connected homogeneous spaces. *Proc. Amer. Math. Soc.*, **1**, 467–469, 1950. (cited on p. 5)

[26] Y. Nikolayevsky and Y. G. Nikonorov. On invariant Riemannian metrics on Ledger–Obata spaces. *Manuscripta Math.*, **158**(3-4), 353–370, 2019. (cited on p. 11)

[27] Yu. G. Nikonorov, E. D. Rodionov, V. V. Slavskii. Geometry of homogeneous Riemannian manifolds. *J. Math. Sciences*, **146**(6), 6313–6390, 2007. (cited on p. 6, 10, 13)

[28] A. L. Onishchik. Topology of transitive transformation groups. Johann Ambrosius Barth Verlag GmbH, Leipzig, 1994. (cited on p. 1, 5)

[29] M. Parton. Old and new structures on products of spheres. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, vol. 288 of *Contemp. Math.*, pp. 406–410. Amer. Math. Soc., Providence, RI, 2001. (cited on p. 7)

[30] F. Podestà. Homogeneous Hermitian manifolds and special metrics. *Transform. Groups*, **23**(4), 1129–1147, 2018. (cited on p. 13)

[31] F. Reidegeld. Spaces admitting homogeneous G_2-structures. *Differential Geom. Appl.*, **28**(3), 301–312, 2010. (cited on p. 2, 5, 13)

[32] S. Salamon. Quaternionic Kähler manifolds. *Invent. Math.*, **67**(1), 143–171, 1982. (cited on p. 6)

[33] F. Tricerri and L. Vanhecke. Naturally reductive homogeneous spaces and generalized Heisenberg groups. *Compositio Math.*, **52**(3), 389–408, 1984. (cited on p. 18)

[34] M. Y. Wang and W. Ziller. Einstein metrics on principal torus bundles. *J. Differential Geom.*, **31**(1), 215–248, 1990. (cited on p. 2)

[35] J. A. Wolf. Spaces of constant curvature. AMS Chelsea Publishing, Providence, RI, sixth edition, 2011. (cited on p. 5)

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